Fano manifolds

Monotone Lagrangians

$L \subset M$

$\mu : \pi_2(M, L) \longrightarrow \mathbb{Z}$ relative c_1

$\omega : \pi_2(M, L) \longrightarrow \mathbb{R}$ $\int u^* \omega$

Monotone L: $\mu = \lambda \omega$ $\lambda > 0$

Coeff. ring

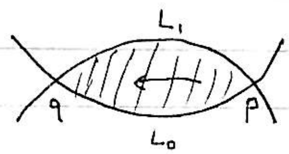
$$\Lambda = \left\{ \sum a_k t^{\nu_k} \mid \nu_k \to \infty,\ a_k \in \mathbb{Z}, \mathbb{C} \right\}$$

Generalisation: Flat line bundles. We will consider objects (L, u),

u = flat ~~line~~ bundle on L.
$U(1)$

$$CF\big((L_0, u_0), (L_1, u_1)\big) = \bigoplus_{p \in L_0 \cap L_1} \mathrm{Hom}\big((u_0)_p, (u_1)_p\big) \otimes \Lambda \ \langle p \rangle$$

δ:



$\cancel{\delta p = t^{\int u^* \omega}}$

$$\delta(h\, p) = t^{\int u^* \omega}\, h' \langle q \rangle$$

$\in \mathrm{Hom}((u_0)_p, (u_1)_p)$ $\in \mathrm{Hom}((u_0)_q, (u_1)_q)$ given by parallel transport along u (contractible, flat bundle $\Rightarrow$ well defined).

If we ~~idor~~ trivialise hom spaces ($\cong \mathbb{C}$, S^1) then $h' = e^{i\theta} h$, u just defines some phase by parallel transport.

So we can just use $CF = \bigoplus \Lambda \langle p \rangle$, but use $\mathbb{C}$ coefficients and count δ with $e^{i\theta}$.

~~for~~ $\langle \delta^2 x, z\rangle \leadsto$ bdry pts of 1-dim moduli space

$\widehat{M}_\beta(x,z) = M_\beta(x,z)/\mathbb{R}$

$\mu(\beta) = 2$

In the $\partial M_\beta(x,z)$ there are strips, disc bundles and sphere bubbles.

μ is at least 2.

$x \neq z$ - at least 1 strip

- all bubbles have $\omega = 1 \Rightarrow$ no discs ($\mu \geq 2$)
- $2c_1$ " no spheres $2c_1 \geq 2$.

$x = z \Rightarrow$ may have disc bubbles, sphere bubbles. at $x = z$.

dim (space of spheres) $= 2n+2-6 = 2n-4$

$\Rightarrow 2n-4+2 < n \Rightarrow$ generically miss x.

Define $m_0(L) = \sum_{\substack{\beta \\ \mu(\beta)=2}}$ # discs in class β passing through x, counted with holonomy of u around $\partial\beta_*$ ($\neq 0$ as disc doesn't lie in L) (and area, but that's constant as $\omega \propto \mu = 2$)

$m_0(L) \in \mathbb{C}$

$0 = \partial \widehat{M} = \langle \delta^2 x, z\rangle + m_0(L_0) - m_0(L_1)$.

Point: for each $\lambda \in \mathbb{C}$ we get Fuk with objects = Lagrangians with $m_0(L_0, u_0) = \lambda$. These categories for $\lambda \neq \lambda'$ are <u>completely</u> separate.

<u>Thm</u>: There is an action of $QH(M)$ on $HF(L,L)$.

Use Morse-Bott picture. Geometric singular chains on L.

$\delta = \partial + \delta'$

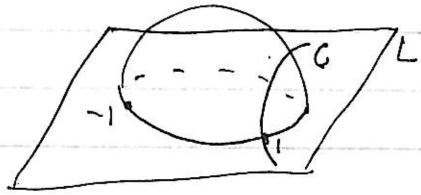
$\delta c = \partial c + \sum_\beta \delta'_\beta c$

$\beta \in \pi_2(M,L)$.

M_β = moduli space of 2 pointed holom. maps (discs w/ boundary on L)

$\delta'_\beta c = ev_{-1\,*}(ev_1^*(c))$

I.e. Constrain $u(1)$ lies on c, look at what $u(-1)$ sweeps out



$n + \mu(\beta) + 2 - 3 = n + \mu - 1$

$(2n + 2c_1)$

$\dim(\delta'_\beta C) = \dim C + (\mu - 1)$

$M_{\beta,L,1}$ - 1 pointed

$[ev_{*1}(M_{\beta,L,1})] \in C_*(L)$

$\parallel$

$m_0[L]$

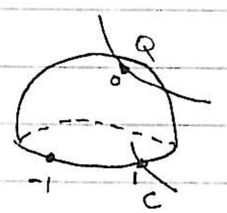
m_2 3-pointed discs

$m_2(C_1, C_2) = \sum_\beta$  $= ev_{3*}(ev_1^* \times ev_2^*)\, C_1 \times C_2$

$Q \in C_*(M)$, $C \in C_*(L)$.

$Q \cap_\beta C$

$M_{-1,0,1}(\beta, L)$



$ev_{-1*}(ev_0 \times ev_1)(Q \times C)$

$Q \cap C = \sum_\beta Q \cap_\beta C$

$\delta(Q \cap C) = \pm\, \partial Q \cap C \pm Q \cap \partial C$ + extra terms that vanish if...

Fano manifold : K_X is ample (divisor = $[D]$)

E.g. $\mathbb{CP}^n$, $c_1 = (n+1)P$

D = union of coordinate $\mathbb{CP}^{n-1}$'s.

On $X \setminus D$ we have a holom. volume form.

<u>Thm:</u> If $m_0(L)$ is not an eigenvalue of

$*c_1(X) : QH \longrightarrow QH$ then $HF(L, L) = 0$

$$\underset{\underset{QH}{\in}}{[c_1(X)]} \cap \underset{\underset{HF}{\in}}{[L]} = \cancel{m_0[L]}\ m_0(L)[L]$$

(there's a proof by pictures)

$$(c_1 - m) \cap [L]$$

If $(c_1 - m_0)*$ is invertible

$$\underbrace{\alpha * ((c_1 - m_0)}_{=[X]} \cap L) = \alpha * 0$$

$$[X] \cap [L] = [L]$$

E.g. $\mathbb{CP}^2$

$$*p : \begin{array}{c|ccc} & 1 & p & p^2 \\ \hline 1 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \\ p^2 & q & 0 & 0 \end{array}$$

eigenvalues for $*c_1 = *3p$

$\lambda^3 - q \cdot 27$

$3q^{\frac{1}{3}}$, $3\sqrt[3]{1}\, q^{\frac{1}{3}}$, $3\sqrt[3]{1}^{\,2}\, q^{\frac{1}{3}}$

Clifford torus : $S^1 \times S^1 \times S^1 \subset \mathbb{C}^3$

$T^2 \subset \mathbb{CP}^2$