Quantum Cohomology

I. J-holomorphic spheres (M^{2n}, ω)

$\pi_2(M) \to H_2(M)$ image = spherical classes.

<u>Thm</u>: $\exists$ a subset $\mathcal{J}_{reg}(\overset{\uparrow}{A}) \subset \mathcal{J}_\tau(M, \omega)$ of second category

spherical class ← tame

s.t. for all $J \in \mathcal{J}_{reg}$, $\mathcal{M}^*(A; J) = \{ \underset{\text{simple}}{u : S^2 \to M} \mid u \text{ J-holom}, [u] = A \}$

is a smooth mfd of dim $= 2n + 2c_1(A)$.

There is an action of $PSL(2, \mathbb{C}) \circlearrowright \mathcal{M}^*(A; J)$ $\Rightarrow$ $\mathcal{M}^*$ is not compact (as $PSL_2 \mathbb{C}$ is not)

<u>Defn</u>:

$$\begin{tikzcd} \mathcal{M}^*(A; J) \times_{PSL_2 \mathbb{C}} (\mathbb{CP}^1)^k \arrow[dd, "ev"] & = \mathcal{M}^*_{0,k}(A; J) \\ & \text{dim} = 2n + 2c_1(A) + 2k - 6 \arrow[u] \\ M^k & \end{tikzcd}$$

<u>Defn</u>. (M, ω) is semipositive if $\forall\ A \in H_2(M)^{\text{sph.}}$, $\omega(A) > 0$, $c_1(A) \geq 3 - n$ $\Rightarrow c_1(A) \geq 0$.

<u>Thm</u>: If (M, ω) semipositive then $\exists\ \mathcal{J}_{reg}(M, \omega) \subset \mathcal{J}_\tau(M, \omega)$ of second category s.t. $\forall\ A \in H_2(M; \mathbb{Z})$, $c_1(A) > 0$, $J \in \mathcal{J}_{reg}(M, \omega)$ then $ev : \mathcal{M}^*_{0,k}(A; J) \to M^k$ gives a pseudocycle of dim $2n + 2c_1(A) + 2k - 6$.

$f : V^d \to M$ s.t. $\dim(\overline{f(V)} - f(V)) \leq d - 2$. $d = \dim \overline{f(V)}$ $(= 2n + 2c_1(A) + 2k - 6)$

<u>Defn</u>: A pseudocycle is

(all these conditions <u>can</u> be removed, it just requires much more technology)

II. 3 pt GW invariants

Notation: $H^k(X) =$ free part of $H^k(X; \mathbb{Z})$

$H^k(X) = \mathrm{Hom}(H_k(X, \mathbb{Z}), \mathbb{Z})$.

$a \in H^i(M)$, $b \in H^j(M)$, $c \in H^k(M)$, $A \in H_2(M)$

$GW^3_A(a, b, c) = ev \cdot (\alpha \times \beta \times \gamma)$ $\qquad (\alpha = PD(a)$ etc.)

$\dim(\mathcal{M}^*_{0,3}) = 2n + 2c_1(A)$

$\Rightarrow$ we require $2n + 2c_1(A) = \deg a + \deg b + \deg c$

Rmk: GW^3_A is graded commutative in a, b, c.

E.g. $A = 0$ - const. maps

$\Rightarrow ev = [\Delta M] \in H_{2n}(M^3)$

$$\Rightarrow GW^3_0(a_1, a_2, a_3) = [\Delta M] \cdot \alpha = \int_M a_1 \cup a_2 \cup a_3$$

$(\mathbb{P}^n, \omega_0)$ $\quad H_2(\mathbb{CP}^n) = \mathbb{Z} = \langle \underset{\nwarrow [\mathbb{CP}^1]}{L} \rangle$

Ex: $c_1(L) = n+1$

$$GW^3_{mL}(a, b, c) = 0 \quad \text{unless} \quad \Sigma \deg = 2n + 2m(n+1)$$

$$\Rightarrow = 0 \quad \text{unless} \quad m = 0 \text{ or } 1.$$

$$GW^3_L(p^i, p^j, p^k) = 1 \iff i + j + k = 2n+1 \qquad (p = PD(L))$$

$$GW^3_L(\underset{\overset{\uparrow}{PD=L}}{p}, \underset{\overset{\nwarrow\ \nearrow}{PD=\text{point}}}{p^n, p^n}) = 1$$

(given 2 pts, 1 line, $\exists$ 1 J-hol sphere hitting all 3).

$$GW^3_{mL}(p^i, p^j, p^k) = \begin{cases} 1 & \text{if } m = 0, \ i+j+k = n \\ 1 & \text{if } m = 1, \ i+j+k = 2n+1 \\ 0 & \text{else} \end{cases}$$

III. Quantum Cohomology

Assume M monotone $\omega(A) = \lambda c_1(A)$ $\lambda > 0$.

As abelian groups, define

$$QH^*(M) = H^*(M) \otimes \mathbb{Z}[q, q^{-1}]$$

Let $N = \min_{c_1(A) \neq 0} |c_1(A)|$. Then $\deg q = 2N$.

Ring structure $a \in H^k(M)$, $b \in H^l(M)$

$$a * b = \sum_{A \in H_2} \underset{\underset{H^{k+l-2c_1(A)}}{\uparrow}}{(a*b)_A} \, q^{c_1(A)/N}$$

where $(a*b)_A$ is defined by

$$\langle (a*b)_A, c \rangle = GW^3_A(a, b, c)$$

(hence $(a*b)_A$ has degree $k + l - 2c_1(A)$).

Note $0 \leq c_1(A) \leq 2n$ $\Rightarrow$ sum is finite.

Extend linearly to QH^* to get $QH^* \otimes QH^* \longrightarrow QH^*$

Rmk: 1) This is distributive

2) graded commutative

3) Thm: This is associative.

Compute $QH^*(\mathbb{CP}^n)$:

$$p^i * p^j = \sum_m (p^i * p^j)_{mL} \, q^{c_1(mL)/N}$$

$$= \sum_{m=\cancel{0}} (p^i * p^j)_{mL} \, q^m$$

$$\langle (p^i * p^j)_{mL}, p^k \rangle = GW^3_{mL}(p^i, p^j, p^k)$$

$$\Rightarrow (p^i * p^j)_{mL} = \begin{cases} p^{\cancel{m}\, i+j} & \text{if } i+j \leq n \\ p^{i+j-\cancel{(}n-\cancel{1}} q & \text{if } \cancel{i+j} \leq n \leq i+j \, \cancel{\leq 2n} \end{cases}$$

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$$\Rightarrow \quad QH^*(\mathbb{P}^n) = \mathbb{Z}[p, q, q^{-1}] \Big/ p^{n+1} = q$$

<u>Sketch of associativity</u>

$$QH^* \otimes QH^* \otimes QH^* \longrightarrow QH^*$$

$$a \otimes b \otimes c \longmapsto (a*b)*c$$

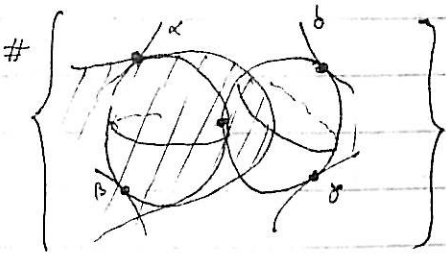
associativity $\Leftrightarrow$ graded commutativity of $\circlearrowleft$

$$(a*b)*c = \pm (b*c)*a = \pm a*(b*c)$$

So: $\langle ((a*b)*c)_A , d \rangle = \langle \sum_B ((a*b)_B \, q^{c_1(B)/N} * c)_A , d \rangle \, q^{c_1(A)/N}$

$$= \langle \sum_B ((a*b)_B * c)_{A-B} , d \rangle \, q$$

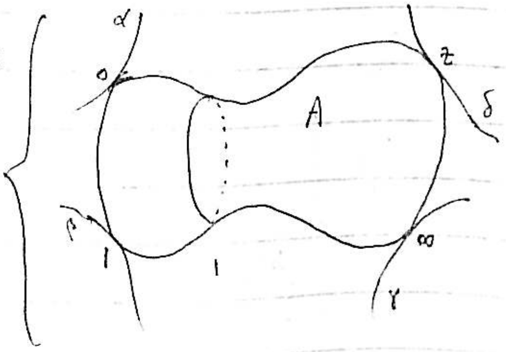
$$= \sum_B GW^3_{A-B} ((a*b)_B, c, d)$$

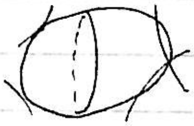
$= \# \Big\{$  $\Big\}$ counting spheres that 'kiss'.

$$= \sum_B GW^{2,2}_{B, A-B} (a, b; c, d)$$

$$= GW_A^{(0,1,\infty,z)} (a, b, c, d)$$

this is graded commutative.

$= \#$ 

Let $z \to \infty$ $\Rightarrow$ count 

Coefficients:

(M, ω) = closed symp.

Λ_ω = Novikov ring of ω

formal sums $\lambda = \sum_{A \in H_2} \lambda(A) e^A$ s.t. $\# \{A \in H_2 \mid \lambda(A) \neq 0, \ \omega(A) \leq c\} < \infty \ \ \forall c \in \mathbb{R}$

$$a * b = \sum_A (a * b)_A \, e^A$$

(M, ω) is CY $c_1 = 0$ on spherical classes

$$\Lambda^\circ = \{ \lambda = \sum_{\xi \in \mathbb{R}} \lambda_\xi t^\xi \mid \# (\xi \in \mathbb{R} \mid \lambda_\xi \neq 0, \ \xi \leq c) < \infty \ \ \forall \xi \in \mathbb{R} \}$$

$$a * b = \sum_A (a * b)_A \, t^{\omega(A)}$$

For general case

$$\Lambda = \Lambda^\circ [q, q^{-1}]$$

$$a * b = \sum_A (a * b)_A \, t^{\omega(A)} q^{c_1(A)}.$$

A lagrangian correspondence induces a map of QH^*, but <u>not</u> as rings.