Talbot 2009

## Combinatorial Fukaya Categories

Setup: $(M, J, \omega, \theta)$

$(M, J)$ = Riemann surface with boundary

$\omega$ = symplectic form (any area form)

$\omega = d\theta$ (exact)

**Preliminary directed Fukaya category** $\mathcal{F}(\mathbb{L})^{pr,\to}$ (no gradings or signs).

**Defn:** Let $\mathbb{L} = (L_1, \dots, L_n)$ be a collection of exact embedded $S^1$'s in $M$, in general position. (transverse, no triple intersections).

Exact $\Leftrightarrow$ $\theta|_{L_i} = df_i$.

N.B. This rules out nullhomologous $L_i$. (~~in fact definition works without exactness assumption as long as $L_i$ are not contractible~~) $\times$ finiteness of $\sigma$

~~The preliminary directed Fukaya category of the $L_i$, over the field $\mathbb{K}$ (where char $\mathbb{K} = 2$) is defined~~

~~$\mathcal{F}(\mathbb{L})^{pr,\to}$ is defined~~

Define the $A_\infty$-category $\mathcal{F}(\mathbb{L})^{pr,\to}$ over field $\mathbb{K}$, char $\mathbb{K} = 2$:

$Ob(\mathcal{F}) = \{L_i\}$

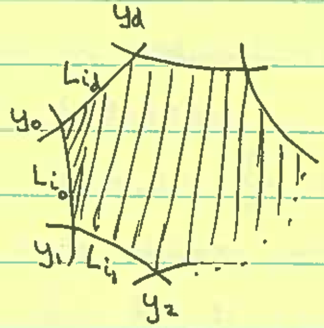
$$\hom(L_i, L_j) = \begin{cases} \mathbb{K}^{|L_i \cap L_j|} \ \text{(basis} = L_i \cap L_j) & \text{for } i < j \\ \mathbb{K} \ \text{(basis} = e_{L_i}) & \text{for } i = j \\ 0 & \text{else} \end{cases} \quad \text{(directed)}$$

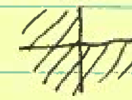
We define the $A_\infty$ structure maps $\mu^d$ as follows:

Given $y_0 \in L_{i_0} \cap L_{i_d}$, $y_k \in L_{i_{k-1}} \cap L_{i_k}$ $1 \le k \le d$, define

$n_d(y_0, \dots, y_d)$ = number of immersed $d$-gons in $M$ as in diagram:

(finite by exactness)



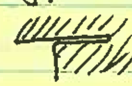

Note: vertices are , not 

~~Then $\mu^d(y_1, \ldots, y_d) := \sum_{y_0} n_d(y_0, y_1, \ldots, y_d)$ if $i_0 < i_1$~~

Then $\mu^d_A : \hom(L_{i_d}, L_{i_{d-1}}) \otimes \ldots \otimes \hom(L_{i_1}, L_{i_0}) \to \hom(L_{i_0}, L_{i_d})$ is defined as

$$\mu^d_A(y_d \otimes \ldots \otimes y_1) = \sum_{y_0 \in \hom(L_{i_0}, L_{i_d})} n_d(y_0, y_1, \ldots, y_d)\, y_0.$$

($= 0$ unless $i_0 < \ldots < i_d$)

(Observations:
- each such polygon ⇝ J-hol disc as in Floer homology, by Riem. mapping theorem
- condition on corners ⇔ 0-dim family, as concave corners can degenerate  )

(Prove $A_\infty$ structure holds)

Coderivation picture: consider the non-unital noncommutative $\mathbb{K}$-algebra

$$T_{\mathbb{K}} = \left( \bigoplus_{\substack{d \geq 1 \\ i_0 < i_1 < \ldots < i_d}} \hom(L_{i_0}, L_{i_1})^* \otimes \ldots \otimes \hom(L_{i_{d-1}}, L_{i_d})^* \quad , \otimes, + \right)$$

Define $\alpha \otimes \beta = 0$ if they don't satisfy correct ordering.

We have the "word-length" decomposition,

$$T = \bigoplus_{n \geq d \geq 1} T_d \qquad \text{(NB. } T_m = 0 \text{ for } m > n\text{)}$$

$T_d$ has a basis as $\mathbb{K}$-vec space given by

$$\{ a_1 \otimes \ldots \otimes a_d \mid L_{i_j} \cap L_{i_{j+1}},\ i_1 < \ldots < i_d \}$$

$a_j = y_j^*$, $y_j \in$ … dual basis

Define the ~~derivation~~ derivation $\delta : T \to T$ by

~~$\delta(T_0) = 0$~~

$$\delta(a_0) = \sum_d n_d(a_0, a_1, \ldots, a_d)\, a_1 \cdots a_d \qquad (= \mu_*^{1*} + \mu_*^{2*} + \mu_*^{3*} + \ldots)$$

(i.e. each polygon contributes its boundary vertices, read anticlockwise).

Note $\delta = \delta_1 + \delta_2 + \ldots$ where $\delta_d : T_1 \to T_d$. ($\delta_0 = 0$ as no 'bubbles', i.e. discs bounded by $L_i$)

Extend $\delta$ to be defined on all $T$ by Leibniz rule.

<u>Prop</u>: $\delta^2 = 0$

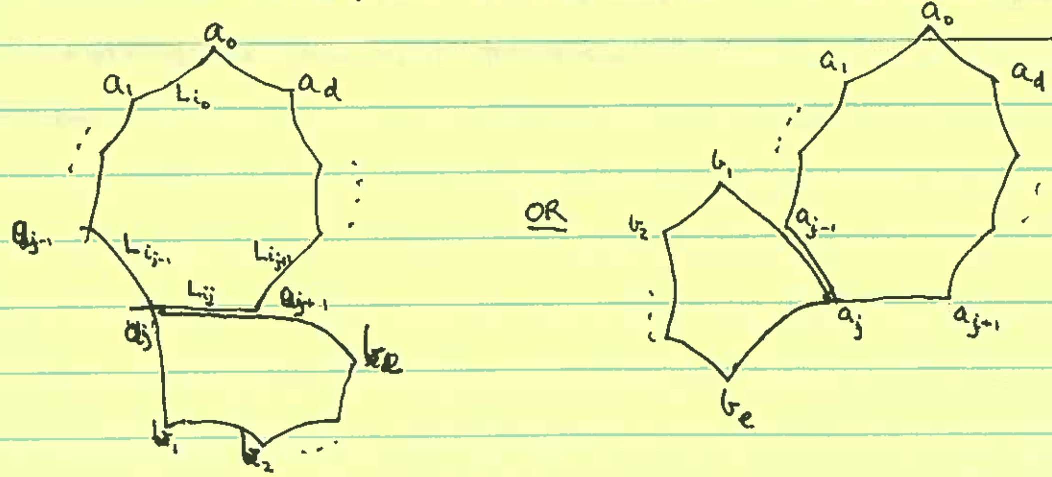
<u>Pf</u>: (In general, this follows by cancelling ends of 1-dim moduli space of polygons, so we imitate that idea).

$$\delta^2 a_0 = \delta\Big( \sum_{\text{polygons}} a_1 \cdots a_d \Big)$$

~~$= \sum_{\text{polygons}} \sum_{\text{polygons}} a_1 \ldots a_{i-1}(b_1 \ldots b_e)\, a_{i+1}$~~

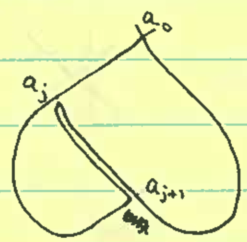
$$= \sum_{\text{polys}} a_1 \ldots \delta a_j \ldots a_d \qquad \text{Leibniz}$$

$$= \sum_{\text{polys}} \sum_{\text{polys}} a_1 \ldots a_{j-1} (b_1 \ldots b_d)\, a_{j+1} \ldots a_d$$

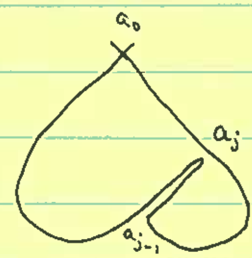


Each term in $\delta^2 a_0$ gives (uniquely) a polygon with one concave corner.

Each such polygon can degenerate in exactly 2 ways to give a term in $\delta^2 a_0$:




or




$\Rightarrow$ terms in $\delta^2 a_0$ cancel in pairs

$\Rightarrow \delta^2 a_0 = 0$

$\Rightarrow \delta^2|_{T_1} = 0$

$\Rightarrow \delta^2 = 0$

Cor: The $\mu^d$ define an $A_\infty$ structure.

(In particular, can define HF, $\mu_2$ etc.)

~~Cor: $(\delta_1 + \delta_2 + \dots)^2 = 0 \Rightarrow \delta_1^2 : T_1 \to T_1$ is $0$~~

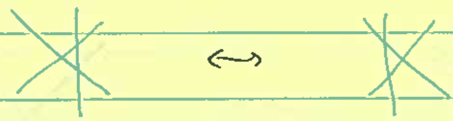
~~$\Rightarrow$ can define homology of $(T_1, \delta)$.~~

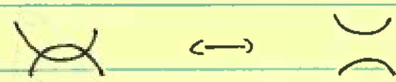
~~This is the direct sum of Lagrangian Floer cohomologies of the Lagrangians $L_i, L_j$, for $i<j$.~~

~~Thm-pt~~

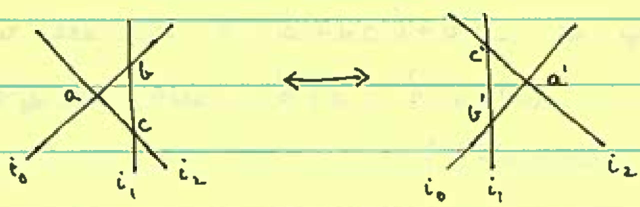
**Prop:** Hamiltonian isotopy of the $L_i$ changes $\mathcal{F}(L_i)^{PT, \to}$ by a quasi-isomorphism of $A_\infty$-algebras.

**Pf:** There are two moves that might come up:

I: 

II: 

We show that each of these changes $\mathcal{F}(L_i)^{PT,\to}$ by a quasi-isomorphism.

I: 


WLOG $i_0$ smallest, by rotating

If $i_0 < i_1 < i_2$ does <u>not</u> hold, then

$a \longleftrightarrow a'$ $\qquad x \longleftrightarrow x'$ for others.

$b \longleftrightarrow b'$

$c \longleftrightarrow c'$

gives a strict isomorphism, because all the polygons are ___ and ▷↔◁ unaffected (the only affected ones are those that run along $i_0, i_1$ and $i_2$).

If $i_0 < i_1 < i_2$, then we define the inverse isomorphisms $f: T \to T'$, $g: T' \to T$

$f: a \longmapsto a' + b'c'$ $\qquad g: a' \longmapsto a + bc$

$x \longmapsto x'$ $\qquad x' \longmapsto x$ (including $x = b, c$)

~~[illegible]~~ $\qquad$ ~~[illegible]~~

Which are also chain maps:

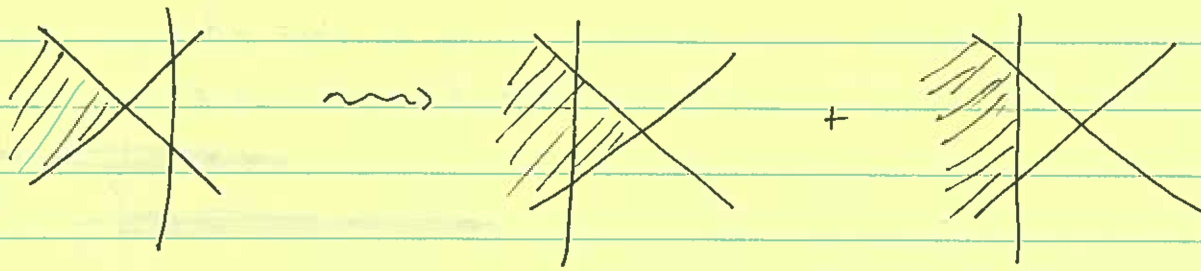
~~for $x \neq a$, $\partial f x = \partial' x$~~

~~$\to \partial x$, but wherever you saw an $a_a$ from the left~~

For $x \neq a$

$$\delta' f x = \delta' x' = f \delta x$$

because wherever you saw an $a$ on the left, you now see $a' + b'c'$:



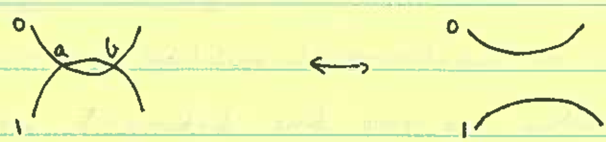
and wherever you saw an $a + bc$ on the right (they always come together) you now see $a' = (a' + b'c') + b'c'$. So you always replace $a$ by $a' + b'c'$. Also $\delta' f a = \delta'(a + bc)$

$$= f(\delta(a + bc))$$

is similar.

Similarly $g\delta' = \delta g$, so $f, g$ are chain maps ($A_\infty$ morphisms) and mutually inverse $\Rightarrow$ strict $A_\infty$ isomorphisms.

II:




Note $\delta a = b + v$. Define $f : T \to T'$

$f : a \mapsto 0$

$b \mapsto v'$

$x \mapsto x'$

This is a chain map because (briefly)


etc.

Furthermore we can define $h_1 : T_1 \to T_1'$ by $h_1(b) = a$, $h_1(x) = 0$ else, and

~~Furthermore we can define~~ $g_1 : T_1' \to T_1$ by

~~$g_1 : x' \mapsto x + n_1(x, b)\, a$.~~ $g_1(x') = x + h_1 \delta_1 x$

This is a chain map, and

$$f_1 g_1 = id$$

$$g_1 f_1 = \delta_1 h_1 + h_1 \delta_1 + id$$

~~where~~ [illegible]

Therefore $f$ is a quasi-isomorphism, hence ~~it has a~~ we can extend $g_1$ to $g : T' \to T$ such that $g$ is a chain map and $f \circ g$, $g \circ f$ are both homotopic to the identity.

$\boxtimes$

~~Cor: The directed Fukaya category is well-defined up to quasi-isomorphism under Hamiltonian isotopy of the Lagrangians. In particular the Floer cohomology is an invariant.~~

Gradings and signs:

By imposing additional conditions on the $L_i$, we can make our $A_\infty$ category $\mathbb{Z}$-graded and over an arbitrary coefficient field $\mathbb{K}$. The extra structure we need is:

($\eta_m$) ① A ~~dist~~ 1-dim distribution $\xi$ on $M$

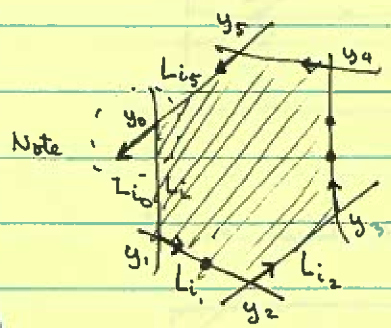
(spin) ② ~~A~~ For each $L_i$, a function $\alpha : L_i \to \mathbb{R}$ so that rotating $\{T_x L_i\}$ by $\alpha(x)$ aligns it with $\xi_x$

($\beta_i$) ③ A ~~real line bundle~~ set of marked points on each $L_i$. ~~corresponding to a real line bundle $\beta \to L_i$, where we choose distinguished marked points~~

(spin) ④ An orientation of each $L_i$.

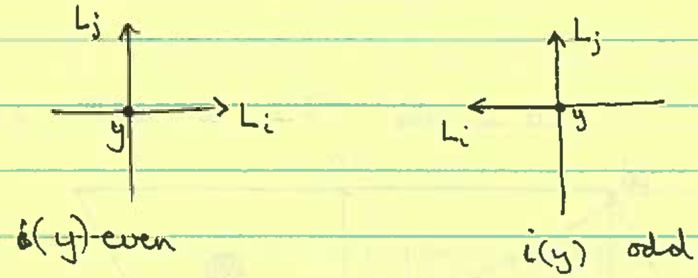
Then we define, for $y \in L_i \cap L_j$, $i < j$, $i(y) = \left\lfloor \frac{\tilde{\theta}(x) - \tilde{\theta}(y)}{\pi} \right\rfloor + 1$

A polygon is counted with sign:




For each vertex $y_i$ if the orientation of $L_{i_j}$ agrees with the arrow (at $y_0$ it's different: compare orientations of $L_{i_2}$) then the sign contribution is $+1$; if it's opposite, the sign contribution is $(-1)^{i(y_j)}$. Each marked point contributes $-1$.

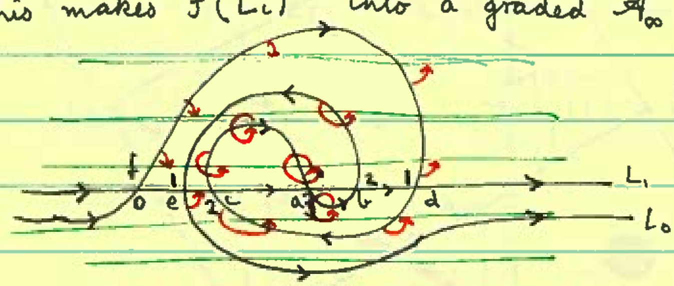
If the distribution $\xi$ is orientable, then we can choose




$(i < j)$.

Then this makes $\mathcal{F}(L_i)^{\rightarrow}$ into a graded $A_\infty$-category, with all the usual degrees of $\mu_d$ etc. (Proofs still work).

E.g.




should have $HF^*(L_0, L_1) = 0$

– check you understand gradings and signs.

- horizontal distribution – work out $\theta(L_0)$ hence gradings
- find bigons, count with sign.

$\delta a = -b + c \qquad \delta f = 0$

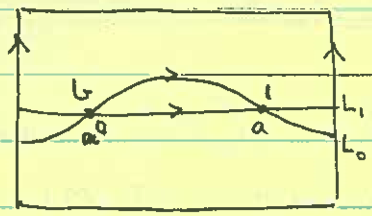
$\delta b = e - d$

$\delta c = e - d$

$\delta d = f$

$\delta e = f$

E.g. PSS isomorphism: $HF^*(L,L) \cong H^*(L)$

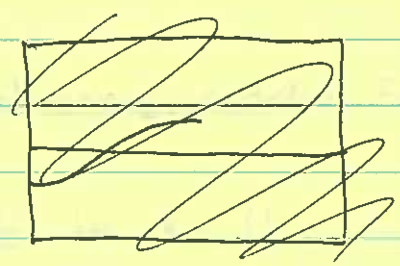



$\xi$ = horizontal

$\delta a = b - b = 0$

$\Rightarrow HF^*(L_0, L_1) \cong \mathbb{Z}\langle a, b\rangle \cong H^*(L_0)$

If we twist one of the pin structures (i.e. $\beta_1$ non-trivial)

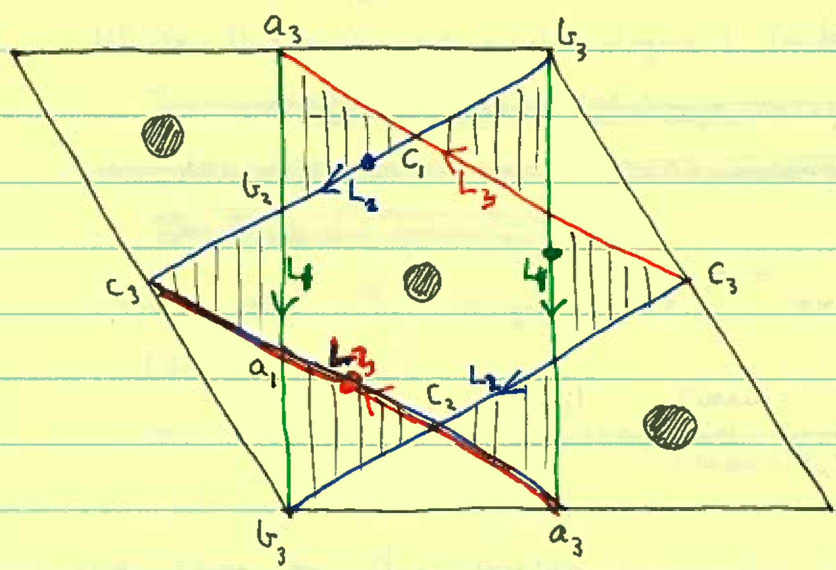


$\rightsquigarrow \delta a = b + b = 2b$

$HF^*(L_0, L_1) \cong \mathbb{Z}_2 = H^*(L_0, \beta)$

(homology with twisted coefficients).

E.g. Mirror to $\mathbb{CP}^2$ (fibre in the...?) Torus with 3 discs removed.




$\delta_2$ is the only non-trivial one. Returning to the $\mu_d$ definition, it defines a product:

$$a_i b_j = \epsilon_{ijk} c_k$$