Product structures on Floer cohomology

Fix (M,ω), for $L_0, L_1 \subset M$ we have

$$HF^*(L_0, L_1) = H^*(CF^*(L_0, L_1), \partial = \mu^1)$$

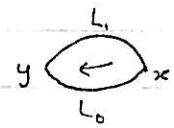
Defn: Assume $L_0 \pitchfork L_1$. Then

$$CF^*(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \Lambda_x$$

$$\Lambda = \left\{ \sum_{k=0}^{\infty} a_k t^{\nu_k} \,\middle|\, a_k \in \mathbb{Z}_2,\ \nu_k \in \mathbb{R},\ \nu_k \to \infty \right\}$$

$$\partial(x) = \sum_y n(x,y)\, y$$

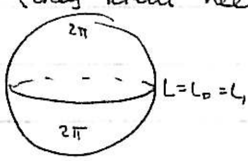
$$n(x,y) = \sum_{\nu \in \mathbb{R}} t^{\nu} n_{\nu}(x,y)$$

where $n_\nu(x,y) = \#$ isolated 

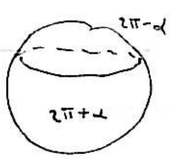
s.t.

$$E(u) = \int_{\mathbb{R}\times[0,1]} u^* \omega = \nu \qquad \left(E(u) = \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial u}{\partial t}\right|^2 ds\, dt.\right)$$

Main property: invariant under exact Lagrangian (i.e. Hamiltonian) isotopy of L_0 or L_1. This requires us to allow negative powers of t in Λ (they aren't needed to define Floer differential).

E.g. 

$$HF^*(L,L) = H^*(S^1, \Lambda)$$



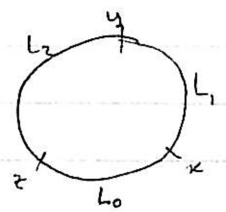
$$HF^*(L,L) = 0.$$

Now take L_0, L_1, L_2 in general position.

$$CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \xrightarrow{\mu^2} CF^*(L_0, L_2)$$

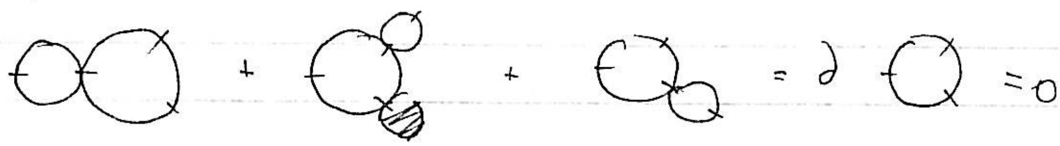
$$(x, y) \longmapsto \sum_z n(x, y, z)\, z$$

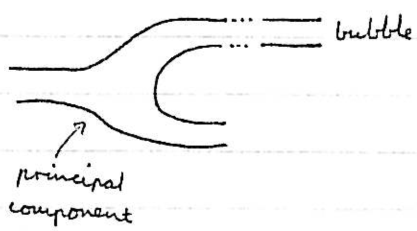
$n(x, y, z)$ counts maps (with energies)



<u>Lem</u>: μ^2 is a chain map

<u>Proof by picture:</u>





Define the Donaldson-Fukaya category

objects $L \subset M$

morphisms $\mathrm{Hom}(L_0, L_1) = HF^*(L_0, L_1)$ (if necessary perturb so they are transverse).

composition induced by μ^2.

* <u>Fact</u>: This category is unital

<u>Fact</u>: Symplectic automorphisms / Hamiltonian isotopy acts (weakly) on this category

This is difficult to get useful information out of. So: pass to the chain level.

Simplified partial version:

Fix a finite ordered collection $(L_1, \ldots, L_m)$ of Lag. submanifolds.

Define the directed Fukaya category $\mathcal{F}^{\rightarrow}(L_1, \dots, L_m)$:

objects $\{L_1, \dots, L_m\}$ in general position

morphisms $\hom(L_i, L_j) = \begin{cases} CF^*(L_i, L_j) & i<j \\ \Lambda \cdot e_{L_i} & i=j \\ 0 & i>j \end{cases}$

This has the structure of an $\mathcal{A}_\infty$ category.

$\mu^1 : \hom(L_i, L_j) \circlearrowleft$ Floer differential $(i<j)$ or zero else

$\mu^2 : \hom(L_j, L_k) \otimes \hom(L_i, L_j) \to \hom(L_i, L_k)$

holomorphic triangle product (+ trivial extensions making e_{L_i} a unit).

...

$\mu^d : \hom(L_{i_{d-1}}, L_{i_d}) \otimes \dots \otimes \hom(L_{i_0}, L_{i_1}) \to \hom(L_{i_0}, L_{i_d})$

nonzero only if $i_0 < \dots < i_d$.

$\mu^d = 0$ for $d > m$.

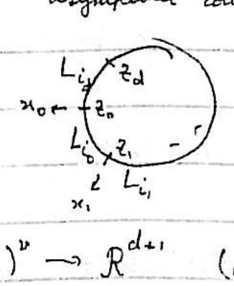
Definition of μ^d: Given $x_1 \in L_{i_0} \cap L_{i_1}$, $x_2 \in L_{i_1} \cap L_{i_2}$, ... and $x_0 \in L_{i_d} \cap L_{i_0}$ (output) and $\nu > 0$, consider

$\mathcal{R}^{d+1}(x_0, \dots, x_d)^\nu = \{ (S, z_0, \dots, z_d, u) \mid$ S = Riem surface isom to closed disc

$z_0, \dots, z_d \in \partial S$ distinct cyclically ordered boundary points

$u : S \setminus \{z_0, \dots, z_d\} \to M$

J-hol map with boundary/asymptotic conditions



and $\int u^*\omega = \nu \}/_{iso}$

This comes with a forgetful map $\mathcal{R}^{d+1}(x_0, \dots, x_d)^\nu \to \mathcal{R}^{d+1}$ (where were the points z_i).

Define $n(x_0, \dots, x_d)^\nu = \# \mathcal{R}^{d+1}(x_0, \dots, x_d) \in \mathbb{Z}/2$

↖ count isolated points, throw rest away.

$$n(x_0, \dots, x_d) = \sum_\nu n(x_0, \dots, x_d)^\nu t^\nu \in \Lambda$$

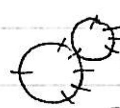
$$\mu^d(x_1, \dots, x_d) := \sum_{x_0} n(x_0, \dots, x_d)\, x_0$$

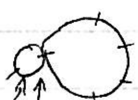
Note $n(x_0, \dots, x_d)$ depend on J (and possibly other auxiliary choices)

Thm: $\mathcal{F}^{\rightarrow}(L_1, \dots, L_m)$ is an A_∞ category.

Pf: Uses compactification of $\overline{\mathcal{R}}^{d+1}(x_0, \dots, x_d)^\nu \longrightarrow \overline{\mathcal{R}}^{d+1}$

$0 = \partial \Big($  $\Big)$ (count boundary points of 1-dim moduli space)

$= \sum$  $e + f = d+1$, $e, f \geq 2$

$+ \sum_i$ 

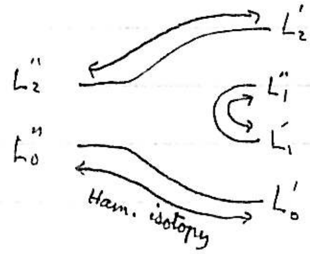
This gives you exactly the A_∞ relations.

Getting an actual A_∞ category $\mathcal{F}(M)$ underlying the Donaldson-Fukaya category.

Involves choices of perturbations: For any pair L_0, L_1, choose L_0', L_1' which are Ham. isotopic, transverse.

$\hom(L_0, L_1) = CF^*(L_0', L_1')$

$\mu^2 : CF^*(L_1'', L_2'') \otimes CF^*(L_0', L_1') \longrightarrow CF^*(L_0''', L_2''')$



Consider maps $u : D^2 \setminus \{z_0, z_1, z_2\} \longrightarrow M$

$u(z) \in \Lambda_z \quad z \in \partial D^2$ (moving boundary conditions)

u is J-holomorphic.

Extend this to all Riemann surfaces that occur in a way consistent with compactification.

Thm: The resulting A_∞ structure is independent of all choices <u>up to quasi-isomorphism.</u>

Works as described when:

1) • $[\omega] = 0$, M not compact but nice at ∞

2) • $[\omega] = \lambda c_1 \quad (\lambda > 0)$

Minimal alg. surf. of gen. type. → 3) • $[\omega] = \lambda c_1 \quad (\lambda < 0)$, $2c_1$ divisible by $n-1$, $n = \dim_{\mathbb{C}} M$

4) • $c_1 = 0$, $n = \dim_{\mathbb{C}} M = 2$ (borderline)

In each case it only works for a particular class of L's.

1) excludes bubbling by Stokes, for L exact.

⋮

4) consider L's with vanishing Maslov class.

No discs for generic J, but then a Lagrangian does not lead directly to a well-defined object of $\mathcal{F}(M)$ (isotoping might hits bubbles).

3) Rules out bubbles:

$$n + 2c_1(A) - 3 \leq -2 \qquad \text{is dim of space of holom. discs}$$

$\Rightarrow$ generically don't exist.