Floer cohomology

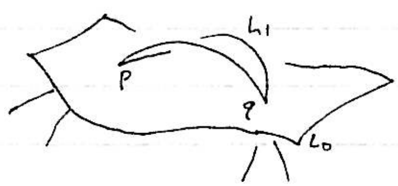
Setting: (M, ω) [$\dim M = 2n$, ω symplectic form]

$L_0, L_1 \subset M$ Lagrangian submanifolds

$L_0 \pitchfork L_1$.

E.g. 1. $M = \mathbb{R}^2$ L_0, L_1 curves in $\mathbb{R}^2$

2. $\dim M = 4$



Floer theory = Morse theory:

① The manifold: $\mathcal{L}M = \{ x \in C^\infty([0,1], M),\ x(0) \in L_0,\ x(1) \in L_1 \}$

Consider the universal cover of $\mathcal{L}M$: $\tilde{x} \in \widetilde{\mathcal{L}M}$ can be written as $(x, [u])$ where $x \in \mathcal{L}M$, $\pi \tilde{x} = x$, u a homotopy from some fixed point $p \in \mathcal{L}M$ to x.

② The function:

$u : [0,1] \times [0,1] \to M$

$\mathcal{A}(x, [u]) = \int_{[0,1]\times[0,1]} u^* \omega$

Ex: $\mathcal{A}$ is well-defined.

③ J = a.c. structure

$T_x \mathcal{L}M = \{$ vector fields X along x s.t. $X(x_0) \in T_{x_0} L_0$, $X(x(1)) \in T_{x_1} L_1 \}$

So $\tilde{g}(X, Y) = \int_0^1 \omega(X(t), JY(t))\, dt$

Ex: ① Critical points of $\mathcal{A}$ are constant paths in $L_0 \cap L_1$

② Gradient flow is given by J-holomorphic maps $u : [0,1] \times \mathbb{R} \to M$

s.t. $u(0, t) \in L_0$, $u(1, t) \in L_1$

$\lim_{t \to \infty} u(s,t) = q$

$\lim_{t \to -\infty} u(s,t) = p$

To define differential, want to know $\dim \mathcal{M}(p,q)$, where
$\mathcal{M}(p,q) = \{\text{flows from } p \text{ to } q\}$

Then we can define the differential,

$$\delta(p) = \sum_{\dim \hat{\mathcal{M}}(p,q) = 0} n_{p,q}\,(q)$$

$$n_{p,q} = \# \hat{\mathcal{M}}(p,q)$$

$$\hat{\mathcal{M}}(p,q) = \mathcal{M}(p,q) / \text{parametrisation}$$

How do we find the dimension of a component of $\mathcal{M}(p,q)$?

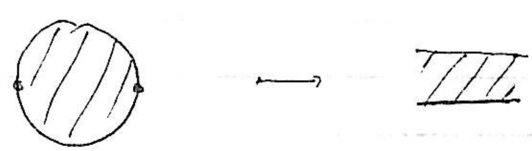
Answer: Maslov Index

Consider $u : (D^2, \partial D^2) \longrightarrow (M, L)$

You can pull back TM to D^2, TL to ∂D^2.

~~Eg. use~~ $u^* TL \subset u^* TM|_{\partial D^2}$ gives a map $S^1 \longrightarrow \Lambda(n)$, $\pi_1(\Lambda(n)) = \mathbb{Z}$
hence we can compute the degree - this is the Maslov index.

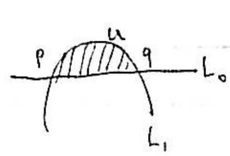
~~Pf~~ Think of a J-hol strip as a J-hol map of the disc into M



Trivialising $u^* TM$ gives $D^2 \times \mathbb{C}^n$.

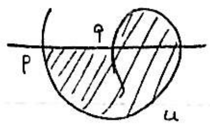
Let $TL_0 \subset TM$ be standard along L_0, then winding along top path is Maslov index.

E.g. In $\mathbb{R}^2$:



~~dim~~ Maslov index $(u) = 1$
$= \dim_u \mathcal{M}(p,q)$.

E.g.

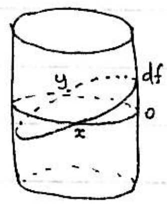


$\dim_u \mathcal{M}(p,q) = 2$ because u has an extra degree of freedom

Take $M = T^*S^1$

L_0 = zero section

L_1 = graph (df)



$\mathcal{M}(x,y)$ has 2 strips

$\mathcal{M}(y,x) = \phi$.

$\Rightarrow \delta x = 2y = 0$ ($\mathbb{Z}_2$ coefficients)

$$\begin{tikzcd} \bullet \arrow[d] \arrow[ddd, bend right=60, "0"'] \\ \mathbb{Z}_2 x \arrow[d, "0"] \\ \mathbb{Z}_2 y \arrow[d] \\ \bullet \end{tikzcd}$$

Take homology: $HF^1(L_0, L_1) = \mathbb{Z}_2$

$HF^0(L_0, L_1) = \mathbb{Z}_2$

<u>Thm:</u> (Floer) Let M be a smooth manifold, f sufficiently (C^2) small. Then

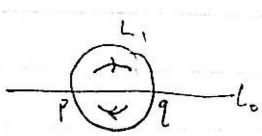
$$\underset{T^*M}{\underset{\cap}{HF^*(\text{0-section}, df)}} \cong H^*(M; \mathbb{Z}_2)$$

<u>Pf:</u> J-holomorphic curves $\leftrightarrow$ gradient flow lines of f.

$\Rightarrow$ Morse theory.

<u>Cor:</u> If $\pi_2(M, L) = 0$, $HF^*(L) = H^*(L)$.

Non-example:



$\delta p = q$ $\quad$ $\delta^2 p = p \neq 0$

$\delta q = p$ $\quad$ $\Rightarrow$ can't define homology.