A_∞ structures in topology

Old: $\cup : H^*M \otimes H^*M \longrightarrow H^*M$

graded comm. assoc. alg.

w/ higher operations (Massey products, Steenrod mod p)

$$\begin{tikzcd} \cup : C^*M \otimes C^*M \arrow[rr] \arrow[dr] & & C^*M \\ & C^*(M \times M) \arrow[ur, "\Delta^*"'] & \end{tikzcd}$$

dga E_∞ algebra.

(E_∞ is like A_∞ but commutative).

M smooth compact oriented d-dim mfd

$\cap : H_{*+d} M \otimes H_{*+d} M \to H_{*+d} M$

"$\cap$": $C_{*+d} M \otimes C_{*+d} M \to C_{*+d} M$

Later: how to see A_∞ and E_∞ structure at level of Morse theory.

Less old (original A_∞ in topology): Pontrjagin product on loops

X ptd space

ΩX based loops

$$(\gamma_1 \circ \gamma_2)(t) = \begin{cases} \gamma_1(2t) & t \le \frac{1}{2} \\ \gamma_2(2t-1) & t \geq \frac{1}{2} \end{cases}$$

<u>Observations:</u>

top. monoid

$$\Omega X \underset{p}{\overset{i}{\rightleftarrows}} \Omega^M X = \{ (r,\gamma) \in \mathbb{R}_{\geq 0} \times \Omega X : \text{if } r = 0,\ \gamma = \text{const} \}$$

$\gamma : [0,r] \longrightarrow X$, $\gamma(0) = \gamma(r) = *$ This <u>is</u> associative

i = inclusion at $r = 1$, p = projection to ΩX

$$\begin{tikzcd} \Omega^\mu X \arrow[r, bend left=50, "id"] \arrow[r, bend right=50, "i \circ p"'] \arrow[r, phantom, "\Uparrow H"] & \Omega^\mu X \end{tikzcd}$$

$$m_2(\gamma_1, \gamma_2) = p\left(i(\gamma_1) \overset{M}{\circ} i(\gamma_2)\right)$$

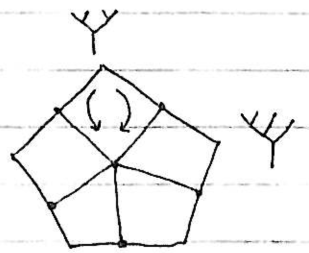
$$m_2(m_2(\gamma_1,\gamma_2), \gamma_3) = p\left(i \cdot p\,(i\gamma_1 \circ i\gamma_2) \circ i\gamma_3\right)$$

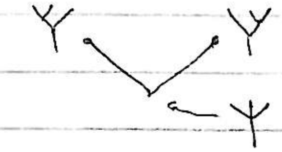
$$\Downarrow$$

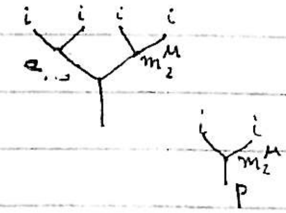
$$p\left(i(\gamma_1) \cdot i(\gamma_2) \cdot i(\gamma_3)\right)$$

$$m_2(\gamma_1, m_2(\gamma_2,\gamma_3)) = p\left(i(\gamma_1) \circ i\,p\,(i(\gamma_2) \circ i(\gamma_3))\right)$$

*







metrised trees with lengths in $[0,\infty]$ give multiplications on ΩX
interior of a cubical subdivision of K_n.

- Easy to transfer strict structures. (? Same as for A_∞ algebras just with trivalent trees?)

<u>Thm</u>: (Recognition Principle) Y has homotopy type of a loop space
$\Leftrightarrow$ Y has homotopy type of a top. monoid with π_0 group
$\Leftrightarrow$ Y has an A_∞ space structure with π_0 a group.

<u>Newer</u>: Loop (String) Product.

$$\begin{array}{ccc} \Omega M & \longrightarrow & LM \\ & & \downarrow \\ & & M \end{array}$$

fibre diagram

const. loops

Serre S.S: $H_{*+d}(M; H_*\Omega M) \Rightarrow H_{*+d}(LM)$

A_∞ alg. $\qquad$ A_∞ alg ?

What about $C_{*+d}(LM)$?

It would be nice to have $H_{*+d} M \to H_{*+d} LM \xrightarrow[\deg -d]{} H_* \Omega M$

$$\begin{tikzcd} \Omega M \arrow[r] \arrow[d] & LM \arrow[d] \\ * \arrow[r] & M \end{tikzcd} \quad \rightsquigarrow H_{*+d} \to H_* \Omega M$$

"intersect cycles w/ ΩM"

$$\begin{tikzcd} & LM \\ LM \times_M LM \arrow[ur] \arrow[r] \arrow[d] & LM \times LM \arrow[d] \\ M \arrow[r, "\Delta"'] & M^2 \end{tikzcd}$$

$$H_{*+d} LM \otimes H_{*+d} LM$$
$$\downarrow \times$$
$$H_{*+2d} (LM \times LM)$$
$$\downarrow \text{"intersect w/ } LM \times_M LM\text{"}$$
$$H_{*+d} (LM \times_M LM)$$
$$\downarrow \text{compare loops}$$
$$H_{*+d} (LM)$$

How do we see $\cup$?

<u>Geometric idea</u> Morse theory:

$[p] \in C_i (M, f) \longleftrightarrow [W^s_f (p)]$ $\nwarrow$ dim $= i$

$[p] \in C^i (M, f) \leftrightarrow PD([W^u_f (p)])$ $\nwarrow$ dim $= d-i$

$$[p] \cup [q] = \sum_{\substack{[r] \text{ crit pt} \\ \mu(r) = \mu(p) + \mu(q)}} \underbrace{([p] \cup [q]) ([r])}_{C_{\mu(r)}} \cdot \underbrace{[r]}_{C^{\mu(r)}}$$

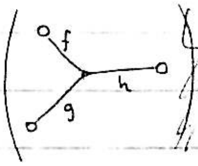
$$= \sum_r \# \left(W^u (p) \cap W^u (q) \cap W^s (r) \right) \cdot [r].$$

Solution: Let f, g, h be "generic" Morse functions

$p \in \mathrm{crit}\, f \quad q \in \mathrm{crit}\, g \quad r \in \mathrm{crit}\, h$

Associative?

~~Fix $f_1, \dots, f_4$ Morse functions.~~

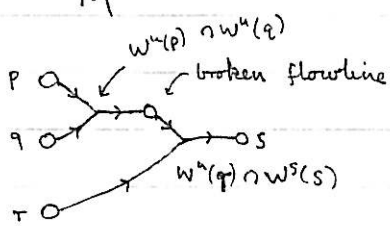
Hom (

, M) "Graph flow" (Morse flow along each edge)

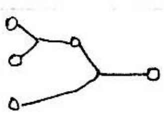
o $\Leftrightarrow$ goes off to ∞.

$\downarrow$ evaluate at middle

M



$$\begin{tikzcd}
\mathring{\mathcal{M}}_{3,0} \times (W^u(p) \cap W^u(q)) \arrow[r] & M \\
F \arrow[u, hook] \arrow[r] & W^u(r) \cap W^s(s) \arrow[u]
\end{tikzcd}$$

Hom ( , M)

$\mathcal{M}_{p,q}$:= moduli space of "metric graphs" with "missing vertices"
(length on compact edges)

+ smooth function on each edge

w/ p numbered univalent edges

q "

$C \subseteq \mathcal{M}_{p,q} \rightsquigarrow$ family cobordism from p points to q points

$$C_*(X)^{\otimes p} \longrightarrow C_*(X)^{\otimes q}$$

1) Closed endpoints, then you can get these maps either H_* or on generic chains

2) Open endpoints

$$\bigotimes_{\substack{i \\ \text{in}}} C_*(X, f_i) \to \bigotimes_{\substack{i \\ \text{out}}} C_*(X, f_i)$$

$\circ \to \bullet$

Change of Morse function (f, g) generic $\circ \to \bullet \to \circ$.