Stasheff polyhedra

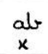
Outline

- Semi-classical polytopes: Stasheff polyhedra
- Moduli space of stable nodal disks (symp. geo./top.)
- Moduli space of metrised ribbon trees (Morse theory)

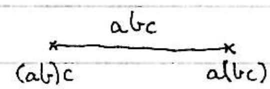
Stasheff Associahedra

Polytopes K_n: K_n has a vertex for each way to meaningfully bracket n variables.

K_2:



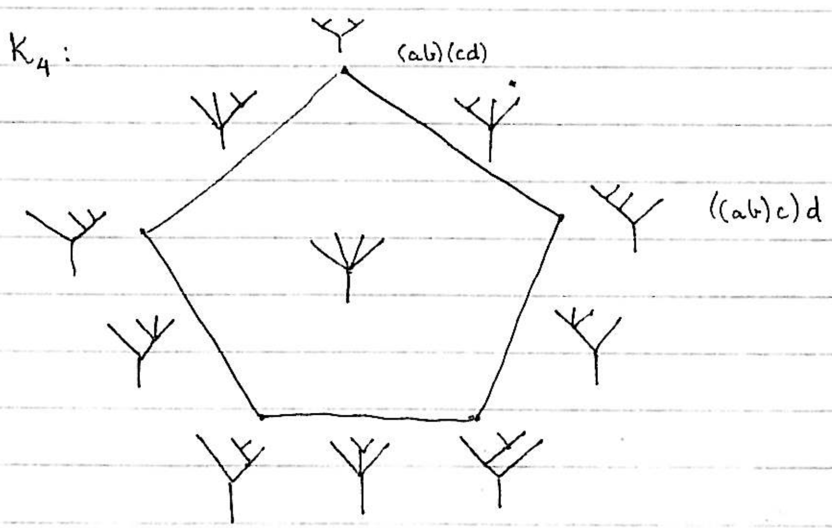
K_3:



...

We can label cells by planar rooted trees with n leaves

K_4:



$\circ_i : K_r \times K_s \hookrightarrow K_{r+s-1}$ graft on ith leaf $1 \le i \le r$

$\{CC_*(K_n)\}_{n\geq 1}$ (cellular chains) form a (non-Σ) operad. i.e. non-symmetric

An algebra over this operad is an $\mathcal{A}_\infty$ algebra.

Motivating example for an operad (non-Σ):

Let X be a set (object in a monoidal category)

$\circ_i : \mathrm{Map}(X^n, X) \times \mathrm{Map}(X^m, X) \longrightarrow \mathrm{Map}(X^{m+n-1}, X)$

$(f, g) \longmapsto f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+m-1}), x_{i+m}, \dots)$

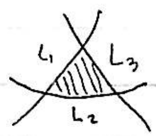
So an operad is a collection of sets $\{P(n)\}_{n\geq 1}$ and $\circ_i$ as above.

The above example is called End_X.

An algebra A over an operad P is a map of operads $P \to \mathrm{End}_A$

i.e. $M_n : P(n) \times A^n \longrightarrow A$.

Suppose we have lagrangians $L_1, L_2, L_3 \subset M$

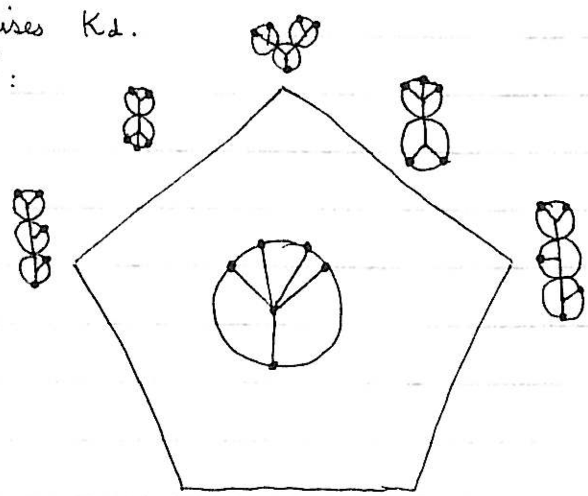


We want to consider moduli spaces of discs with $(d+1)$ marked points $\mathcal{R}^{d+1}$. We have to compactify (á la Deligne-Mumford)

$$\overline{\mathcal{R}^{d+1}} = \coprod_{\substack{\text{trees, at least trivalent}\\ \text{with } d \text{ leaves}\\ \& \text{ a root}}} \mathcal{R}^T$$

This realises K_d.

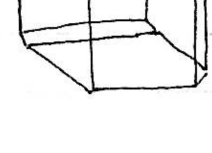
E.g. $\overline{\mathcal{R}^{4+1}}$:



$\mathring{K}_n = \mathrm{int}(K_n)$

$K_n = \coprod_{\text{trees } T} \mathring{K}_n^T$

$\mathring{K}_n^T = \prod_{\text{vertices}} \mathring{K}_{|v|-1}$

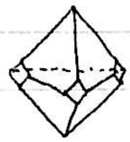


comes from this?

*

K_4 ~~Assoc~~ has a 5-fold symmetry from rotating labeling of disc

K_5 has 6-fold:



$\underset{\text{rotation}}{\mathbb{Z}_3} \times \mathbb{Z}_2$ ← reflection

Thm (Stasheff): $K_d = \overline{\mathcal{R}^{d+1}} \approx I^{d-2}$

2) FO / Paul 9.2

$\overline{\mathcal{R}^{d+1}}$ is a smooth $(d-2)$-mfd w/ corners (in particular so-and-so # edges come together at vertex, etc.)

Metrised Ribbon Trees

Defn: A ribbon tree is a ~~planar~~ pair (T, i) where T is a tree, $i: T \hookrightarrow D^2 \subset \mathbb{C}$ embedding s.t.

1) No vertex of T has 2 edges
2) $v \in T$ has one edge $\Rightarrow i(v) \in \partial D^2$
3) $i(T) \cap \partial D^2$ = one-edge vertices.

Identify $(T, i) \sim (T', i')$ if T, T' are isomorphic, i and i' isotopic.

Defn: G_k = set of triples (T, i, v_1), $v_1 \in i(T) \cap \partial D^2$, $|i(T) \cap \partial D^2| = k$.

Fix $t \in G_k$, let $Gr(t)$ be the set of all maps $l: C^1_{int}(T) \to \mathbb{R}^+$ (like a metric on your graph)

↑ interior edges of T

Defn: $Gr_k = \bigcup_{t \in G_k} Gr(t)$.

Prop: $Gr_k \approx \mathbb{R}^{k-3}$

Goal: 1) Build moduli spaces out Gr_k and $\overline{\mathcal{R}^{d+1}}$ to do Morse theory and Lagrangian intersection theory.

Let (M, g) be a Riem. mfld and choose $f_1, \dots, f_k \in C^\infty(M)$ s.t. $\{f_{i+1} - f_i\}$ are Morse functions ($f_{k+1} = f_1$).

An element of $M_g(M : \bar{f}, \underset{\overset{\shortparallel}{(p_1, \dots, p_k)}}{\bar{p}})$ is $((T, i, v_1), \ell) \in Gr_k$ and $I : T \to M$ s.t.

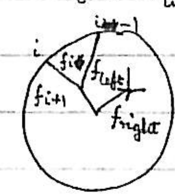
1) I continuous

$I(v_i) = p_i$

Metrise T s.t. $e \in C^1_{ext}(T) \Rightarrow e$ isometric to $(-\infty, 0]$, and $e' \in C^1_{int}(T) \Rightarrow e'$ isometric to $[0, \ell(e')]$.

2) $\dfrac{\partial I|_{e_i}}{\partial t} = -\operatorname{grad}(f_{i+1} - f_i)$, $e_i \in C^1_{ext}$

3) $\dfrac{\partial I|_{e_i'}}{\partial t} = -\operatorname{grad}(f_{left(e_i')} - f_{right(e_i')})$ $\quad e_i' \in C^1_{int}$



Thm: (F-O) For generic $\{f_i\}$

$\mathcal{M}_g(M; \bar{f}, \bar{p})$ is a C^∞ mfd.

$M_J(T^*M, \bar{\Lambda}^\varepsilon, \bar{X}^\varepsilon)$ $\quad (*) \quad\quad X^\varepsilon \in T^*M$

Λ_i^ε is the graph of $\varepsilon df_i \subset T^*M$ is Lagrangian, J compatible a.c. structure.

p_i is a critical point of $f_{i+1} - f_i \Rightarrow X_i^\varepsilon = (p_i, \underset{\underset{\varepsilon df_{i+1}(p_i)}{\shortparallel}}{\varepsilon df_i(p_i)})$

$\Rightarrow$ lies in intersection $\Lambda_i^\varepsilon \cap \Lambda_{i+1}^\varepsilon$

An element of $(*)$ is a pair $([z_1, \dots, z_k], w)$, $[z_1, \dots, z_k] \in \mathbb{R}^{k-1+1}$

$w : D^2 \to T^*M$

1) $w(z_i) = X_i^\varepsilon$

2) $w(\partial_i(D^2)) \subset \Lambda_i$ $\quad$ Here $\partial_i(D^2) = i$th boundary component between z_i, z_{i+1}.

3) J-holomorphic.

"Thm": (F-O) $\mathcal{M}_g(M : \bar{f}, \bar{p}) \cong M_J(T^*M, \Lambda_i^\varepsilon, X_i^\varepsilon)$ are diffeomorphic as smooth manifolds for ε small enough. Note this depends on $k \Rightarrow$ can't choose same ε to work for all k.