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<u>Lagrangian Submanifolds</u>

(M, ω) $(2n)$-dim symplectic manifold

ω = nondegenerate closed 2-form

$[\omega] \in H^2(M; \mathbb{R})$

L = n-dim submanifold

$i : L \longrightarrow M$

$i^* \omega = 0$

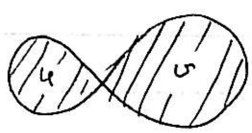
i = immersion $\Rightarrow$ <u>immersed Lagrangian submanifold</u>

embedding $\Rightarrow$ <u>embedded</u> " "

If $\omega = -d\alpha$ is exact, $i^*\alpha = dH$, L is called <u>exact</u>.

Gromov: In $(\mathbb{C}^n, \omega_0)$, any embedded compact Lagrangian submanifold is <u>not</u> exact.

E.g. The immersed lagrangian in $\mathbb{C}$:



is exact if and only if area(u) = area(v).

<u>Defn</u>: $\varphi : [0,1] \times L \longrightarrow M$ is a lag. isotopy if

$\varphi^* \omega = dt \wedge \alpha + \cancel{\beta}$ (i.e. β vanishes)

$j_t : L \hookrightarrow [0,1] \times L$ inclusion $j_t(x) = (t, x)$

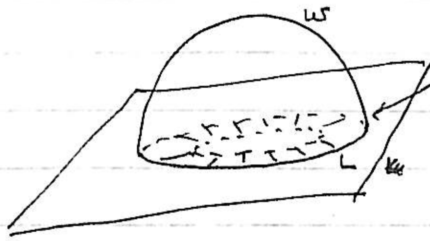
$j_t^* \alpha$ closed 1-form

<u>exact</u> lagrangian isotopy: $j_t^* \alpha$ is exact $\Leftrightarrow$ $j_t^* \alpha = dH$ (= hamiltonian isotopy).

<u>Defn</u>: Given (M, L) and $w : (D^2, \partial D^2) \longrightarrow (M, L)$

$$I_\omega(w) = \int_{D^2} w^* \omega$$

$I_\omega(w)$ is invariant under exact lagrangian isotopy of L



area swept out by ~~to~~ exact lag. isotopy vanishes.

<u>Defn</u>: (Maslov class)

$[\mu] \in H^2(M, L; \mathbb{Z})$

$w : (D^2, \partial D^2) \longrightarrow (M, L)$

$$\begin{tikzcd} (w^* TM, w^* TL) \arrow[r] \arrow[d] & (TM, TL) \arrow[d] \\ (D^2, \partial D^2) \arrow[r, "w"] & (M, L) \end{tikzcd}$$

$\exists$ symplectic trivialisation:

$$\begin{tikzcd} (\mathbb{C}^n, \mathbb{R}^n) \arrow[r, "\Phi"] \arrow[dr] & (w^* TM, w^* TL) \arrow[d] \\ & (D^2, \partial D^2) \end{tikzcd}$$

$\Phi^* w \big|_{\partial D^2}$ is a loop (call it γ) in the Lagrangian Grassmannian $\Lambda(n)$.

$$\Lambda(n) = \{ A \cdot \mathbb{R}^n \mid A \in U(n) \} /_{\sim} \quad = U(n) /_{O(n)}$$

$$A_1 \mathbb{R}^n = A_2 \mathbb{R}^n$$
$$A_1 A_1^t = A_2 A_2^t$$

Given a loop $\gamma : S^1 \longrightarrow \Lambda(n)$, we can define
$\mu(\gamma) = \deg(\det A A^t)$.

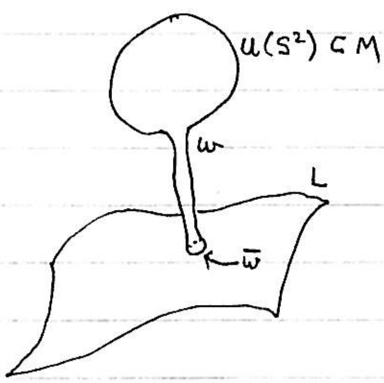
$I_\mu(w)$ depends only on the homotopy class of w, so it defines a map $I_\mu : \pi_2(M,L) \longrightarrow \mathbb{Z}$.

Prop: Given
$$w : (D^2, \partial D^2) \longrightarrow (M,L)$$
$$\bar{w} : (D^2, \partial D^2) \longrightarrow (M,L)$$
$$w|_{\partial D^2} = \bar{w}|_{\partial D^2}$$
we can define
$$u : S^2 \longrightarrow M$$
$$\shortparallel$$
$$D^2 \# -D^2$$

Then $I_{\mu,L}(w) - I_{\mu,L}(\bar{w}) = 2\, c_1(u)$. $(= 2\, c_1(TM)\cdot[u])$

~~move w and $\bar{w}$ so $\bar{w}^* TM$~~

Hence, if we take a 2-sphere u and connect sum of it with a contractible disc with boundary on L, the Maslov class is $2c_1(u)$



Defn: $L \subset M$ is a <u>monotone</u> Lagrangian submanifold if it is Lagrangian and
$I_\omega = \lambda\, I_{\mu,L}$ $\quad \lambda \geq 0$ const.

A symplectic manifold M is called monotone if
$[\omega] = \lambda [c_1]$

By the above observation (about the 2-sphere), L monotone Lagrangian $\Rightarrow M$ monotone.

Let $L \hookrightarrow M$ be immersed / embedded lag. in (M, ω)

$$\begin{array}{l} T^*L \\ \downarrow \\ L \end{array} \qquad \begin{array}{l} \omega_{can} = -d\,\alpha_{can} \\ \theta \in \Omega^1(L) \Rightarrow \theta^* \alpha_{can} = \theta \quad \text{(defines } \alpha_{can}\text{)} \end{array}$$

Darboux - Weinstein Thm: Given such L, there exists a nbhd $\mathcal{U}$ of L in T^*L and an immersion / embedding $\Phi : \mathcal{U} \to M$

the zero section

Such that

$$\Phi^* \omega = \omega_{can}$$

$$i \circ \Phi = \text{zero section.}$$

Examples of Lagrangian submanifolds:

(1) If $\theta \in \Omega^1(L)$ is closed then $\theta(L) \subset T^*L$ is Lagrangian.

(2) Another example:

Suppose (M, ω) symplectic

$$\sigma : M \to M$$

$$\sigma^* \omega = -\omega \quad \text{(antisymplectic)}$$

$$\sigma^2 = \mathrm{Id}.$$

Then $L = \mathrm{Fix}(\sigma)$ is Lagrangian (e.g. $\mathbb{RP}^n \subset \mathbb{CP}^n$).

~~If $I_\omega = 2\lambda$~~,

If $I_\omega = 2\lambda I_{c_1}$, then L is a monotone lag. submanifold.

(3) Clifford torus in $\mathbb{CP}^n$.

$$T^{n+1} = \underbrace{S^1(1) \times \dots \times S^1(1)}_{n+1} \hookrightarrow S^{2n+1}(1) \hookrightarrow \mathbb{C}^n$$

(here $S^1(1)$ = unit sphere in $\mathbb{C}$)

Then

$$T^{n+1}/_{S^1} \cong T^n \subset \mathbb{C}^n/_{S^1} = \mathbb{CP}^n$$

This T^n is a monotone Lagrangian submanifold.

$$\to \pi_2(T^n) \to \pi_2(\mathbb{CP}^n) \xrightarrow{i} \pi_2(\mathbb{CP}^n, T^n) \underset{\xleftarrow{\;j\;}}{\xrightarrow{\partial}} \pi_1(T^n) \xrightarrow{} \pi_1(\mathbb{CP}^n) \to \cdots$$

$$\Rightarrow \quad \pi_2(\mathbb{CP}^n, T^n) \cong \pi_2(\mathbb{CP}^n) \oplus \underbrace{\pi_1(T^n)}_{n \text{ generators } w_1, \ldots, w_n.}$$

$$w_k(z) = [1:1:\ldots:1:\underset{k+1}{z}:1:\ldots:1] \qquad \text{(for } z \in S^1\text{, i.e. } |z|=1 \text{ in } \mathbb{C}\text{).}$$

$\mu(w_k) = 2$

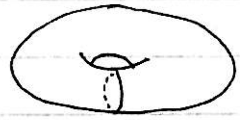
(because $\mu\left(\begin{bmatrix} 1 \ldots e^{2\pi i t} & 0 \\ 0 & \ddots 1 \end{bmatrix} \cdot \mathbb{R}^n\right) = 2$)

$w_0(z) = [z:1: \ldots :1]$

$\mu(w_0) = 2.$

~~Therefore~~ Each w_i gives an element of $\pi_2(M, L)$ via

$j(w_0)(z) = [z:1:\ldots:1]$ etc., $z \in D^2 \subset \mathbb{C}$



Note $\partial\left([w_0] + \ldots + [w_n]\right) = 0$ (j from L.E.S. at top of page)

$\cancel{\Rightarrow [w_0] + \ldots + [w_n] = i([\alpha])}$ $\Rightarrow [w_0] + \ldots + [w_n] = i([\alpha])$ by exactness.

$I_\mu\left(j([w_0] + \ldots + [w_n])\right) = 2(n+1)$ (each has Maslov index 2)

$\cancel{I_\omega\left(j([w_0] + \ldots + [w_n])\right)}$

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(4) Standard tori in $\mathbb{C}^n$

$$T^n_{a_1,\dots,a_n} = S^1(a_1) \times \dots \times S^1(a_n) \hookrightarrow \mathbb{C}^n$$

If $a_1 = \dots = a_n = a$ it is monotone with $\lambda = \frac{\pi a^2}{2}$

(5) Chekanov tori in $\mathbb{C}^n$.

$$S^1 \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$$

$$(t, x_1, \dots, x_n) \longmapsto (e^{x_1}\cos(2\pi t),\ e^{x_1}\sin(2\pi t),\ x_2, \dots, x_n)$$

$$(i^*)^{-1} : T^*(S^1 \times \mathbb{R}^n) \longrightarrow T^*\mathbb{R}^{n+1} \cong \mathbb{C}^{n+1}$$

If L is embedded lagrangian in $\mathbb{C}^n$:

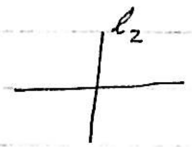
$$\Theta_a(L) := (i^*)^{-1}\left((a\text{-section of } S^1) \times L\right)$$

This is an embedded lagrangian in $\mathbb{C}^{n+1}$ for $a \in S^1$.

L is monotone in $\mathbb{C}^n$ $\Rightarrow$ $\Theta_0(L)$ monotone in $\mathbb{C}^{n+1}$.

<u>Lagrangian surgery</u>

(V^{2n}, ω) symplectic. Suppose we have



Suppose $f : S^{n-1} \times \mathbb{R} \hookrightarrow \mathbb{C}^n$

$$f(S^{n-1} \times [c, \infty)) = l_1 \setminus B_1 = \text{disc in } l_1$$

$$f(S^{n-1} \times (-\infty, -c]) = l_2 \setminus B_2 = \text{disc in } l_2$$