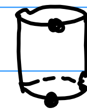
Projective plane

We saw $\mathbb{A}^1 \longleftrightarrow$ 

$\mathbb{P}^1 \longleftrightarrow$ 

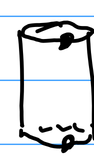
this time: $\mathbb{P}^2 \longleftrightarrow$?

Rather than give a defn of 'stops' in higher dimensions (for which see [Sylvan], [Ganatra-Pardon-Shende]), we'll give an alternative formulation.

<u>Defn</u>: A <u>Landau-Ginzburg (LG) model</u> is a pair (X, W), where X = cpx mfd, $W : X \longrightarrow \mathbb{C}$ holomorphic function. (called the <u>superpotential</u>).

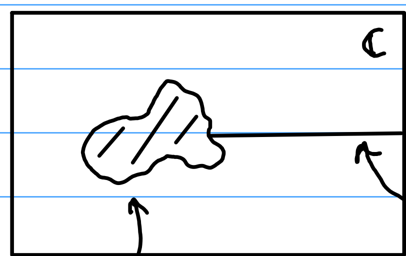
A LG model gives rise to a stopped symplectic mfd $\left(|W| \leq R,\ \sigma = W^{-1}(-R)\right)$. Thus we have

$\mathbb{A}^1 \longleftrightarrow (\mathbb{C}^*, W = z) \rightsquigarrow$ 

$\mathbb{P}^1 \longleftrightarrow (\mathbb{C}^*, W = z + z^{-1}) \rightsquigarrow$ 

<u>**Defn:**</u> The <u>Fukaya-Seidel category</u> $FS(X, W)$ has

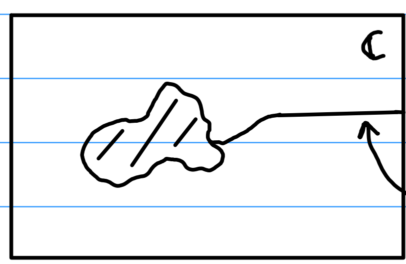
$Ob =$ exact $L \subset X$ such that $W(L)$ looks like



fibration over $\mathbb{R}_+$ near ∞

can do anything it wants in a compact region

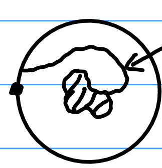
$$\hom(L_0, L_1) := \mathbb{C}\langle L_0 \cap \varphi(L_1)\rangle$$



$W(\varphi(L))$

fibration over $\mathbb{R}_+ + i\varepsilon$ near ∞

(compare:



wrap as close as possible to stop

stop at $-R$)

Compositions similar to before.

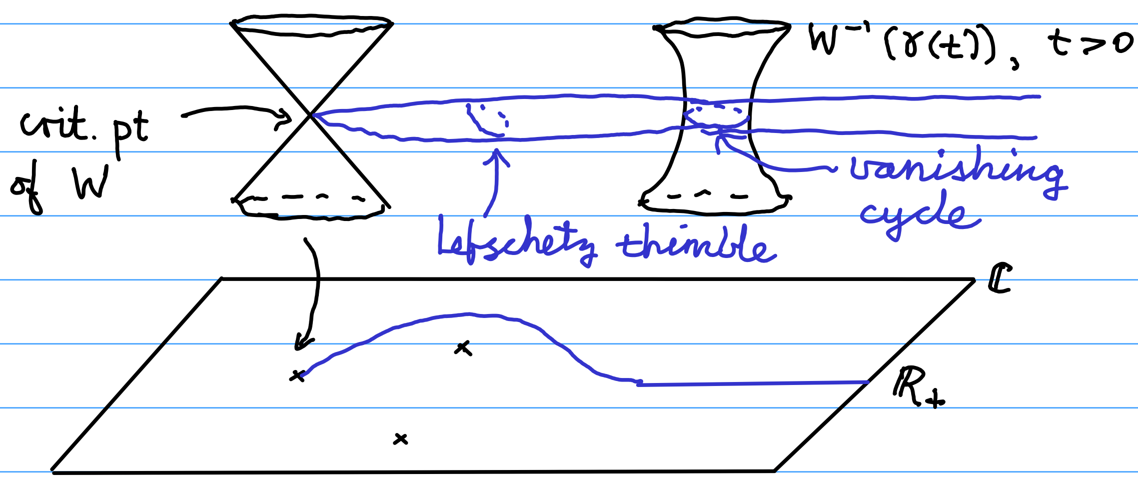
This time we're going to study

$$D^b\,\mathrm{Coh}(\mathbb{CP}^2) \simeq FS\left((\mathbb{C}^*)^2,\ W = x+y+\frac{1}{xy}\right).$$

So far we only saw how to compute Fukaya categories in dim $=2$ – hard to describe J-hol. curves explicitly if we don't have Riem. mapping thm.
Happily, we can reduce this FS cat to a computation in dim 2.

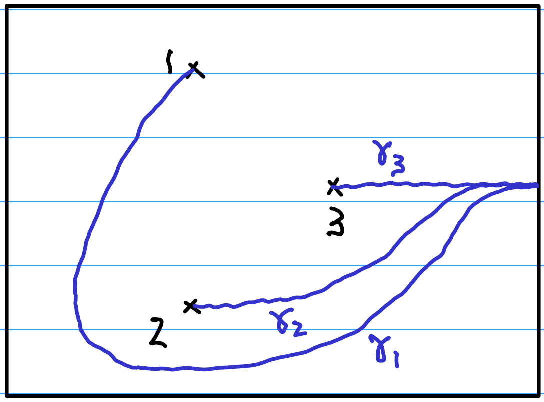
Suppose W is a Lefschetz fibration: fibres of W are symplectic submanifolds, and singularities are as simple as possible: modelled on $W(\underline{z}) = \sum_i z_i^2$. (Also need control at ∞ – see [Seidel, Vanishing cycles...]).

Given a 'vanishing path' $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{C}$ avoiding critical vals of W, $\gamma(t) = t$ for $t \gg 0$, there's a Lefschetz thimble lying over γ:



It defines an object L_i of $FS(W)$.

Choose vanishing paths γ_i for each crit. value, disjoint away from $\mathbb{R}_{>>0}$, 'entering $+\infty$ in order $\gamma_1, \gamma_2, \ldots$':



Check: $\hom(L_i, L_j) = 0$ if $j < i$

In fact, the L_i form an <u>exceptional collection</u> (Seidel). Furthermore, we can compute the category purely inside a fibre of W:

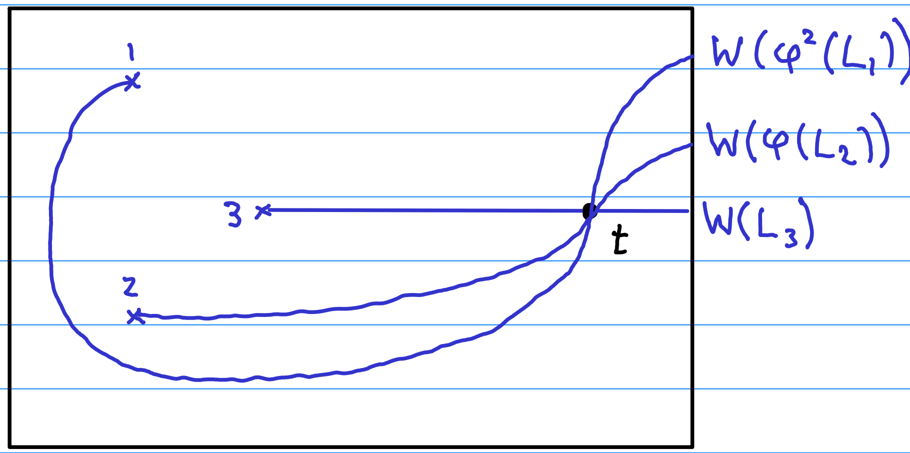
<u>Prop</u>: Let $F := W^{-1}(t)$ for some $t >> 0$.
Define $V_i := L_i \cap F$ (vanishing cycles).
Then

$$\hom_{FS(W)}(L_i, L_j) = \begin{cases} 0 & \text{if } j < i \\ \mathbb{C} \cdot e & \text{if } i = j \\ \hom_{Fuk(F)}(V_i, V_j) & \text{if } i < j \end{cases}$$

Furthermore,

$$m_k^{FS(w)}(x_1, \ldots, x_k) = m_k^{Fuk(F)}(x_1, \ldots, x_k).$$

Pf: We arrange



Then we have identifications

$$L_i \cap \varphi(L_j) = V_i \cap V_j$$

Furthermore, if u is a holomorphic disc with boundary on L_i, then $W \circ u$ is a holom. disc with boundary on γ_i; the only such disc is the constant one at t; so u is contained in F. $\square$

F is 2-dim'l, so we can compute $m_k^{Fuk(F)}$ with Riem. mapping theorem. This computation, and its mirror, is explained in [Seidel, More about VC..., Section 3]. He describes 3 van. cycles for the mirror to $\mathbb{P}^2$ and computes homs between them. They correspond to the objects $\underset{L_0}{\mathcal{O}}$, $\underset{L_1}{\Omega^1(1)}$, $\underset{L_2}{\Omega^2(2)}$ of $D^b\mathrm{Coh}(\mathbb{P}^2)$.

The homs between them are

$$\hom(L_i, L_j) \simeq \Lambda^{j-i}(\mathbb{C}^3)$$

with composition given by multiplication in the exterior algebra.

(Seidel expresses this in terms of a quiver).

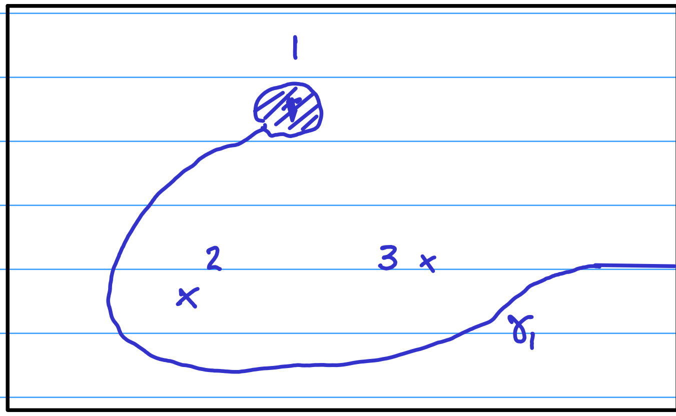
<u>Remark:</u> Suppose W is not a Lefschetz fibration. Let c_i = crit. val. (W), and choose vanishing paths γ_i, disjoint away from ∞ as before. Let

$$FS_i \subset FS$$

be the subcategory with objects

$$Ob(FS_i) := \{ L : W(L) \in \mathrm{nbhd}(\gamma_i) \}.$$

(if W is a Lefschetz fibration then the only interesting object of FS_i is L_i).



Then we have

$$\hom^*(L_i, L_j) = 0 \quad \text{if } L_i \in FS_i,\ L_j \in FS_j,\ i > j$$

by the same argument as before.

In nice cases these subcategories 'generate', and we have what's called a 'semi-orthogonal decomposition'

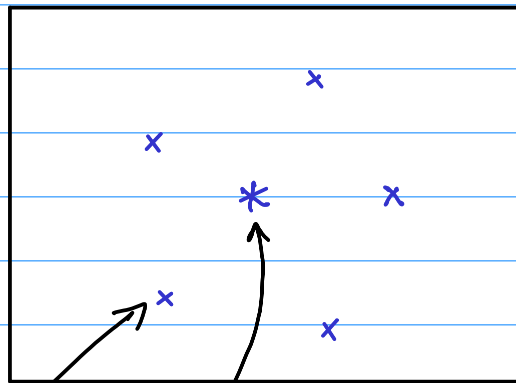
$$FS = \langle FS_1, \dots, FS_n \rangle.$$

As an example of this, let Y be a deg-d hypersurface in $\mathbb{CP}^n$. If $d < n$ Y is Fano, and

$$D^b\mathrm{Coh}\, Y = \langle \mathcal{K}_Y, \mathcal{O}, \dots, \mathcal{O}(n-d) \rangle$$

Kuznetsov category

The Hori-Vafa mirror is defined in [She16]. It is $([\mathbb{C}^n/G], W)$, where crit. val. of W are:



complicated singularity at 0 (K_Y)

Lefschetz-type singularities at $e^{2\pi i k/n-d}$ $(\mathcal{O}(k))$.