Introduction to the Fukaya category

1. Cartoon

(X, ω) = symp. mfd

$\mathrm{Fuk}(X, \omega) = A_\infty$ cat.

$(\omega|_L = 0,\ \dim L = \frac{1}{2} \dim X)$

Objects = Lagrangian submanifolds $L \subset X$
+ extra conditions, data ⟵ 'brane structure'

$$\hom(L_0, L_1) = K\langle L_0 \cap L_1 \rangle$$

$$m_k : \hom(L_0, L_1) \otimes \dots \otimes \hom(L_{k-1}, L_k) \longrightarrow \hom(L_0, L_k)$$

$$m_k(p_1, \dots, p_k) := \sum_{p_0,\, u \in \mathcal{M}_0(p_0, \dots, p_k)} e^{-2\pi \omega(u)} \cdot p_0$$

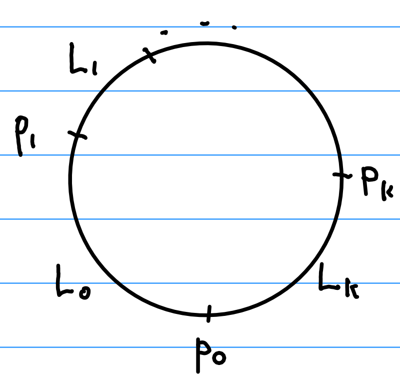
0-dimensional component

where

$$\mathcal{M}(p_0, \dots, p_k) := \{ u : \Sigma \longrightarrow X \text{ holomorphic} \} / \sim$$

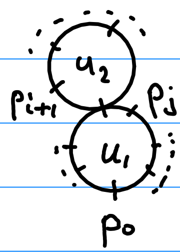
disc with
k+1 boundary
marked pts z_i

$u(z_i) = p_i$
$u(z) \in L_i$
for $z \in \partial \Sigma$
between z_i, z_{i+1}



These satisfy the A_∞ relations because

$$\sum_{i,j} \pm m_*(p_1, \ldots, m_*(p_{i+1}, \ldots), p_{j+1}, \ldots, p_k)$$

$$= \sum_{p_0}$$ 

$$\cdot \, e^{2\pi i(\omega(u_1)+\omega(u_2))} \cdot p_0$$

$\partial \overline{\mathcal{M}}_1(p_0, \ldots, p_k)$

and (signed) # boundary points of an oriented compact 1-manifold is 0.

What is a 'holomorphic' map $u: \Sigma \longrightarrow X$?
One satisfying

$$J \circ Du = Du \circ j$$

$J \in$ End(TX), almost-cpx structure on X: $J^2 = -\mathrm{Id}$, not nec. integrable.

$j \in$ End$(T\Sigma)$, complex structure on Σ

We require J to be ω-compatible, meaning $\omega(\cdot, J\cdot) =$ Riem. metric. (E.g., J is integrable, ω a Kähler form). This ensures that

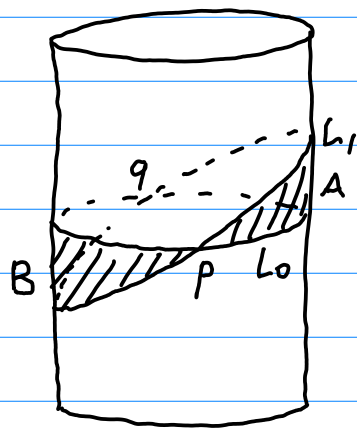
$$\omega(u) = \int_\Sigma \|du\|^2 \geqslant 0 \text{ with equality} \iff u \text{ constant.}$$

$\|du\|^2$ ← "energy"

Deforming J changes Fuk(X, ω) by a quasi-equivalence.

The space of ω-compatible J is contractible, so $\mathrm{Fuk}(X, \omega)$ is well-defined up to q.eq.

E.g.

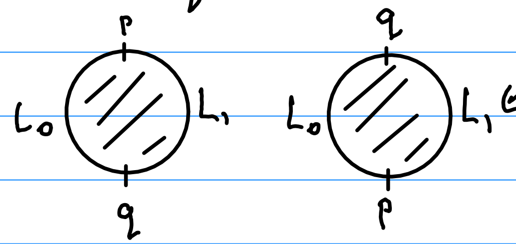


$$\hom(L_0, L_1) \simeq \mathbb{C}\cdot p \oplus \mathbb{C}\cdot q$$

$$m_1(p) = \left(e^{-2\pi A} - e^{-2\pi B}\right)\cdot q$$

$$m_1(q) = 0$$

two of these



none of these (orientations don't match)

The Riemann mapping theorem makes computation of Fukaya categories essentially combinatorial in this dimension.

Result: $$\mathrm{Hom}(L_0, L_1) \simeq \begin{cases} \mathbb{C}\cdot p \oplus \mathbb{C}\cdot q & \text{if } A = B \\ 0 & \text{if } A \neq B. \end{cases}$$

2. The lies that were told; towards fixing them

Q: What's the '0-dimensional component of $\mathcal{M}(p_0, \dots, p_k)$'?

A: Consider:

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ \mathcal{M} & \subset & C^\infty(\Sigma, X) \end{array}$$

$\uparrow$ topologise $\mathcal{M}$

$$E_u := \mathrm{Hom}(T\Sigma, u^* TX)$$

$$\cup$$

$$E_u^{1,0} := \{ \xi : J \circ \xi = \xi \circ j \}$$

We have a section D of E: $u \mapsto Du$.

$u \in \mathcal{M} \iff Du \in E^{1,0}$.

Claim: If Du is <u>transverse</u> to $E^{1,0}$ at u, then a nbhd of $u \in \mathcal{M}$ is an $i(u)$-dim'l mfld, where $i(u)$ depends only on homotopy class of u.

This uses index theory and a version of the implicit function theorem for infinite-dim'l spaces (Smale).

If transversality holds at all $u \in \mathcal{M}$, then

$\{u \in \mathcal{M} : i(u) = i\} = i$-dim'l mfld.

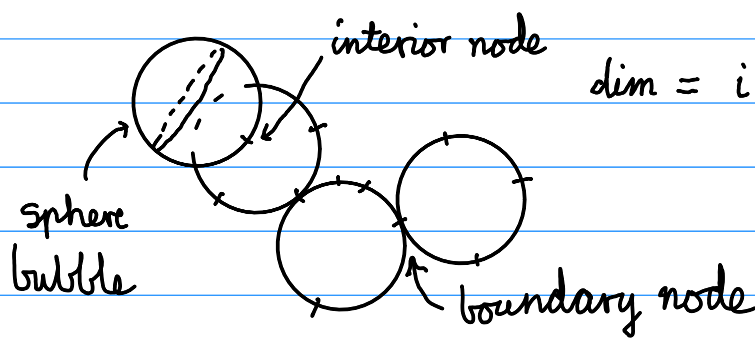
$$\mathcal{M} = \coprod_i \mathcal{M}_i$$

In general transversality need not hold for any J, but there are ways around this, working in varying degrees of generality (domain-dependent perturbations, Kuranishi structures, polyfolds, implicit atlases...).

Q: $\mathcal{M}_0(p_0, \ldots, p_k)$ may be infinite, why does the sum defining m_k converge?

A: It may not! To see how to fix this, need <u>Gromov compactness</u>. This concerns a partial compactification $\mathcal{M}(p_0, \ldots, p_k) \subset \overline{\mathcal{M}}(p_0, \ldots, p_k)$ by <u>nodal</u> discs:

Gromov topology



dim $= i(u) - \#$ boundary nodes $- 2\#$ interior nodes

It says the part of $\overline{\mathcal{M}}$ consisting of curves of energy $\leq E$ is <u>compact</u>. The idea is that non-compactness can only be caused by energy 'concentrating' at points, which may be in the interior or on the boundary.

Now, the dimension formula says all strata of $\overline{\mathcal{M}}_i - \mathcal{M}_i$ have dimension $< i$; if $i = 0$ this means they're empty, so $\overline{\mathcal{M}}_0 = \mathcal{M}_0$.

Gromov compactness then says, in the sum defining m_k, there are finitely many terms with $\omega(u) \leq E$. This doesn't guarantee convergence, but suggests the remedy.

Defn: $\Lambda := \left\{ \sum_{i=0}^{\infty} a_i \cdot T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = +\infty \right\}$.

(Novikov field)

This will be the coefficient field of $\mathrm{Fuk}(X, \omega)$, not $\mathbb{C}$.

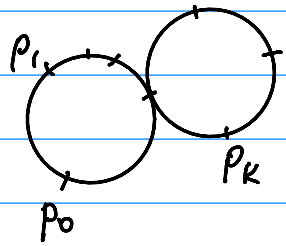
So, set $\quad \hom(L_0, L_1) := \Lambda \langle L_0 \cap L_1 \rangle$

and

$$m_k(p_1, \dots, p_k) := \sum_{u \in \mathcal{M}_0(p_0, \dots, p_k)} T^{\omega(u)} \cdot p_0 .$$

This infinite sum is well-defined, by Gromov compactness. Note $T^\lambda = \exp(\lambda \log T)$, so we can think of this as a 1-parameter family of categories $\mathrm{Fuk}(X, -\frac{\log T}{2\pi} \cdot \omega)$, in a formal nbhd of $T = 0$.

Note: Gromov compactness is also used to prove A_∞ rel'ns: it shows that $\overline{\mathcal{M}}_1^E(p_0, \dots, p_k)$ (E = energy) is a compact 1-mfd with boundary

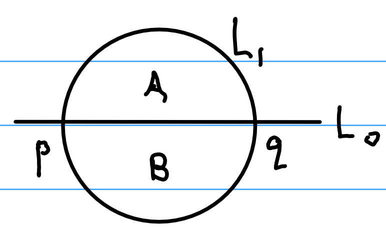


$\longleftrightarrow$ coefficient of $T^E \cdot p_0$ in $\sum \pm m_*(p_1, \dots m_*(p_i \dots) \dots p_k)$.

Q: A_∞ categories have m_k with $k \geq 1$; why can't Fuk have m_0, counting Ⓠ ?

A: It can! In this case we say our A_∞ cat. is <u>curved</u>. One approach is to impose a topological constraint on L, e.g. $\omega : \pi_2(X, L) \to \mathbb{R}$ vanishes, as part of the requirement to be a 'brane'; this ensures that all J-holomorphic discs with ∂ on L have zero energy, hence are constant, forcing $m_0 = 0$.
More generally, we allow $m_0 \neq 0$, but equip L with a bounding cochain 'deforming m_0 to 0'.
When we study Fano varieties we'll see another possibility

E.g.



$\hom(L_0, L_1) = \mathbb{C}\cdot p \oplus \mathbb{C}\cdot q$

$m_1(p) = T^A \cdot q$

$m_1(q) = T^B \cdot p$

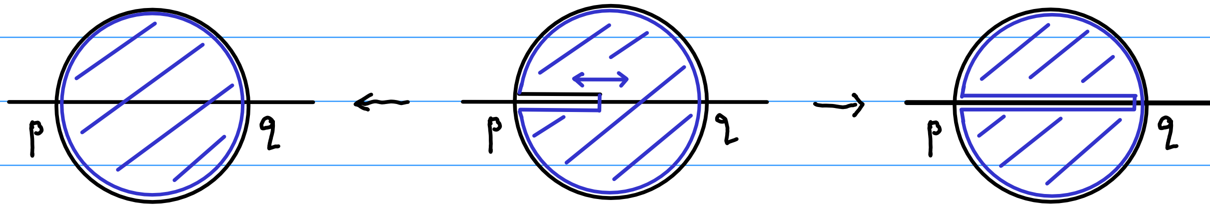
$m_1(m_1(p)) = T^{A+B} \cdot p \neq 0 \Rightarrow A_\infty$ rel'ns don't hold

Explanation: $m_1(m_1(p)) - m_2(p, m_0^{L_1}) = 0$ curved A_∞ rel'n

where $m_0^{L_1} = T^{A+B} \cdot \underset{\text{unit}}{e_{L_1}}$ arises from the disc bounded by L_1, and

$$m_2(p, T^{A+B} \cdot e_{L_1}) = T^{A+B} \cdot p$$

What went wrong with the proof of $m_1^2 = 0$?
Consider $\widetilde{\mathcal{M}}_1(p, p)$:



$m_2(p, m_0)$ $\qquad\qquad$ $m_1(m_1(p))$

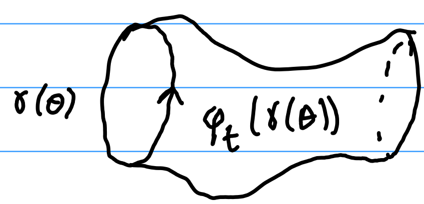
Q: What if L_0 is not transverse to L_1 (e.g., $L_0 = L_1$)?

A: We'll perturb L_1 by a <u>Hamiltonian isotopy</u>.

<u>Defn</u>: Let $\varphi_t : [0,1] \times X \to X$ be a 1-param. family of symplectomorphisms, i.e. $\varphi_t^* \omega = \omega \ \forall t$. Define

$$\mathrm{Flux}(\varphi_t) \in H^1(X; \mathbb{R})$$

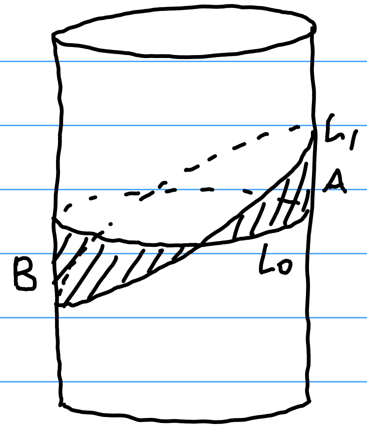
by $$\mathrm{Flux}(\varphi_t)(\gamma) := \int_{\varphi_t(\gamma(\theta))} \omega$$



φ_t is called a <u>Hamiltonian isotopy</u> if $\mathrm{Flux}(\varphi_t) = 0$.

<u>Claim</u>: If φ_t is a Hamiltonian isotopy, then the objects L, $\varphi_1(L)$ are isomorphic in $H^*(\mathrm{Fuk})$.

E.g.



L_1 Ham. isot. to L_0

$\Leftrightarrow$ $A = B$

$\Leftrightarrow$ $\mathrm{Hom}(L_0, L_1) \neq 0$.

Using the claim (which we do not prove), one can give a definition of $H^*(\mathrm{Fuk})$, and even of Fuk, by setting

$$\mathrm{hom}(L_0, L_1) := \mathrm{hom}(L_0, \varphi(L_1))$$

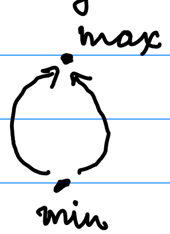
for some auxiliary Hamiltonian isotopy making $L_0 \pitchfork \varphi(L_1)$.

<u>Lem</u>: If $\omega : \pi_2(X, L) \longrightarrow \mathbb{R}$ vanishes, L compact, then

$$\mathrm{Hom}^*(L, L) \simeq H^*(L)$$

as graded algebras.

This is proved by relating the holomorphic strips to Morse flowlines (e.g. the two strips we saw in the cylinder correspond to flowlines



on L).

We'll see more detail when we study the 'Morse-Bott model' for $\mathrm{Hom}^*(L, L)$ later; take it on faith for now.

Q: Why is it $\mathbb{Z}$-graded?

A: Part of the 'brane structure' associated with L is an algebro-topological datum called a 'grading'. It allows us to associate $i(p) \in \mathbb{Z}$ to $p \in L_0 \cap L_1$ (a kind of winding number). We define

$$\hom^i(L_0, L_1) := K\langle p \in L_0 \cap L_1 : i(p) = i \rangle.$$

We have a formula

$$i(u) = i(p_0) - i(p_1) - \dots - i(p_k) + 2 - k$$

$\Rightarrow$ since m_k comes from counting u with $i(u) = 0$, we have

$$i(p_0) = i(p_1) + \dots + i(p_k) + 2 - k$$

$\Rightarrow$ m_k has degree $2-k$.

Q: Signs?

A: The other part of the brane structure is a spin structure on L. This determines an orientation of $\mathcal{M}(p_0, \dots, p_k)$.

We can also replace '$\mathbb{C}$' with '$\mathbb{Z}/2\mathbb{Z}$' in the definition of Λ.