1. What is MS?

Cpx mfd $= (X, J)$ $J \in \mathrm{End}_{\mathbb{R}}(TX)$, $J^2 = -\mathbb{1}$, integrable

smooth mfd

Symp mfd $= (X, \omega)$ $\omega \in \Omega^2(X, \mathbb{R})$, $\omega^{\wedge \mathrm{top}}$ nowhere-vanishing, $d\omega = 0$

Kähler mfd $= (X, \omega, J)$, $\omega(\cdot, J\cdot) =$ Riem. metric.

A compact Kähler is called Calabi-Yau if $\Omega^{\mathrm{top}}_X \simeq \mathcal{O}_X$ (canonical bundle is trivial).

String theory associates a 'theory' $t(X, \omega, J)$ to a CYK. There's an involution m on the set of such theories. MS says there should exist a 'mirror' $(X^\vee, \omega^\vee, J^\vee)$ such that

$$m(t(X, \omega, J)) = t(X^\vee, \omega^\vee, J^\vee).$$

This theory is too hard to define mathematically. Witten defined 'topological twists'

$$\begin{tikzcd} & t(X, \omega, J) \arrow[dl, "\text{A-twist}"'] \arrow[dr, "\text{B-twist}"] & \\ A(X, \omega) & & B(X, J) \end{tikzcd}$$

indep't of J indep't of ω

m interchanges A- and B- twists. Thus:

$$\begin{tikzcd} A(X,\omega) \arrow[dr, leftrightarrow] & A(X^\vee,\omega^\vee) \arrow[dl, leftrightarrow] \\ B(X,J) & B(X^\vee,J^\vee) \end{tikzcd}$$

These topologically twisted theories can be defined mathematically. They are 'open-closed topological conformal field theories'.

There are forgetful maps:

$$\begin{tikzcd} \text{OCTCFT's} \arrow[r, shift left] & A_\infty \text{ categories} \arrow[l, shift left] \arrow[r, shift left, "\text{'take cohomology'}"] & \text{categories} \arrow[l, shift left, "\times"] \end{tikzcd}$$

reconstruction theorem in nice cases (Costello)

The A_∞ category associated to:

- $A(X,\omega)$ is $DFuk(X,\omega)$ = <u>derived Fukaya category</u>.

- $B(X,J)$ is $DCoh(X,J)$ = <u>derived category of coherent sheaves</u>.

Thus (in nice cases) MS is equivalent to:

<u>Conj</u> (Homological MS): $\exists$ A_∞ quasi-equivalences

$$\begin{tikzcd} DFuk(X,\omega) \arrow[dr, leftrightarrow] & DFuk(X^\vee,\omega^\vee) \arrow[dl, leftrightarrow] \\ DCoh(X,J) & DCoh(X^\vee,J^\vee) \end{tikzcd}$$

Taking cohomology gives corresponding equivalences of <u>categories</u> = easier to understand. We will mostly work on the level of categories. However note that Costello's reconstruction theorem only works for the <u>A_∞</u> categories.

Plan: 1) Define $DFuk(X, \omega)$ (next time)

2) Define $DCoh(X, J)$

3) Study a different mirror pair $(X, X^\vee)$ each week, and see the equivalence of categories concretely.

2. (A_∞) categories

Defn: A <u>K-linear category</u> consists of

1) A set of objects
2) For each pair of objects, a K-vector space $\mathrm{Hom}(X,Y)$
3) k-linear 'composition maps'

$$\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$$

$$a \otimes b \longmapsto a \cdot b$$

satisfying <u>associativity</u>

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

4) For each object X, an element

$$e_X \in \mathrm{Hom}(X,X)$$

satisfying

$$e_X \cdot a = a = a \cdot e_Y$$

for all $a \in \mathrm{Hom}(X,Y)$.

A $\mathbb{Z}$-graded k-linear category is one where the Hom-spaces are $\mathbb{Z}$-graded, and $|a \cdot b| = |a| + |b|$.

Defn: A **k-linear A_∞ category** consists of:

1) A set of objects

2) For each pair of objects, a $\mathbb{Z}$-graded vector space $hom^\bullet(X,Y)$

3) k-linear 'composition maps'

$$m_k : hom^\bullet(X_0, X_1) \otimes \cdots \otimes hom^\bullet(X_{k-1}, X_k) \to hom^\bullet(X_0, X_k)$$

for $k \geq 1$, of degree $2-k$, satisfying the **A_∞ relations**:

$$\sum_{i,j} (-1)^{t} m_{k+1-j}(x_1, \ldots, x_i, m_j(x_{i+1}, \ldots, x_{i+j}), x_{i+j+1}, \ldots, x_k) = 0.$$

where $t = |x_1| + \cdots + |x_i| + i$.

'Taking cohomology':

The $k=1$ A_∞ relation says

$$m_1(m_1(x)) = 0 \iff m_1 \text{ is a differential.}$$

Define $Hom^*(X,Y) := H^*(hom^*(X,Y), m_1)$.

$k=2$ says

$$m_1(m_2(x,y)) + m_2(m_1(x), y) + (-1)^{|x|+1} m_2(x, m_1(y)) = 0.$$

$\Rightarrow$ the operation

$$\mathrm{Hom}^*(X,Y) \otimes \mathrm{Hom}^*(Y,Z) \to \mathrm{Hom}^*(X,Z)$$

$$[x] \otimes [y] \longmapsto [x]\cdot[y] := (-1)^{|x|} [m_2(x,y)]$$

is well-defined (Leibniz-type rule). It has degree $2-2=0$.

$k=3$ says

$$m_2(m_2(x,y),z) + (-1)^{|x|+1} m_2(x, m_2(y,z))$$

$$+ m_1(m_3(x,y,z)) + m_3(m_1(x),y,z) + \dots = 0$$

$$\Rightarrow ([x]\cdot[y])\cdot[z] = [x]\cdot([y]\cdot[z])$$

Therefore we have defined a $\mathbb{Z}$-graded K-linear category with the same set of objects as our original category - except for the unit axiom! We say the A_∞ cat. is <u>cohomologically unital</u> if its cohomological category satisfies the unit axiom. (All our A_∞ categories will be c-unital.)

E.g. $R = K$-algebra

$Ch(R)$ is an A_∞ cat. with

Obj = chain complexes of $\overline{R}$-modules

$$\cdots \longrightarrow M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots$$

$$d^{i+1} \circ d^i = 0$$

$$\hom^j((M^\bullet, d^\bullet_M), (N^\bullet, d^\bullet_N)) := \prod_i \mathrm{Hom}_R(M^i, N^{i+j})$$

$$m_1(f) := f \circ d_M - (-1)^{|f|} d_N \circ f$$

$$m_2(f, g) := g \circ f$$

modulo signs (sorry)

$m_{\geq 3} = 0$ (this kind of A_∞ cat. is called a differential graded (dg) cat.)

Ex: The A_∞ relns hold (those with ≥ 4 inputs are trivial).

This is an example of the kind of category that shows up on the B-side of HMS. The A-side is the Fukaya category, which we'll introduce next time.

Rmk: The Fukaya category is not dg. How can HMS match it with a dg category?

<u>Defn</u>: An <u>A_∞ functor</u> $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of:

- a map $F : \mathrm{Ob}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{D}$
- K-linear maps
  $F^k : \hom^{\cdot}_{\mathcal{C}}(c_0, c_1) \otimes \dots \otimes \hom^{\cdot}_{\mathcal{C}}(c_{k-1}, c_k) \longrightarrow \hom^{\cdot}_{\mathcal{D}}(Fc_0, Fc_k)$
  of degree $1-k$, for $k \geq 1$, satisfying

$$\sum_{i,j} (-1)^{\dagger} F^{k+1-j}(c_1, \dots, m_j^{\mathcal{C}}(c_{i+1}, \dots), c_{i+j+1}, \dots, c_k)$$

$$= \sum_{j, i_1, \dots, i_j} m_j^{\mathcal{D}}(F^{i_1}(c_1, \dots), F^{i_2}(c_{i_1+1}, \dots), \dots, F^{i_j}(\dots, c_k))$$

where $\dagger = |c_1| + \dots + |c_i| + i$.

<u>Ex</u>: F determines a K-linear (ordinary) functor

$$H^*(F) : H^*(\mathcal{C}) \longrightarrow H^*(\mathcal{D}).$$

<u>Defn</u>: A_∞ cats $\mathcal{C}$ and $\mathcal{D}$ are <u>quasi-equivalent</u> if $\exists$ A_∞ func. $F : \mathcal{C} \longrightarrow \mathcal{D}$ s.t. $H^*(F)$ is an equivalence.

It is perfectly possible for a dg and non-dg A_∞ cat. to be quasi-equivalent.

3. OCTCFT

Defn: An open-closed topological conformal field theory consists of

1) A vector space C called the space of closed strings.

2) A set, whose elements are called branes;

3) For each pair of branes, a vector space $O(\lambda, \lambda')$ called the space of open strings from λ to λ'.

4) Maps between these spaces of open and closed strings will come from Riemann surfaces Σ with boundary, equipped with

   - a set $P_{int} = P_{int}^{in} \sqcup P_{int}^{out}$ of interior punctures;

   - a set $P_\partial = P_\partial^{in} \sqcup P_\partial^{out}$ of boundary punctures;

   - a labelling of the components of $\partial\Sigma \setminus P_\partial$ by D-branes.

Specifically, if $\mathcal{M}$ denotes the moduli space of Riemann surfaces of this topological type, we have maps

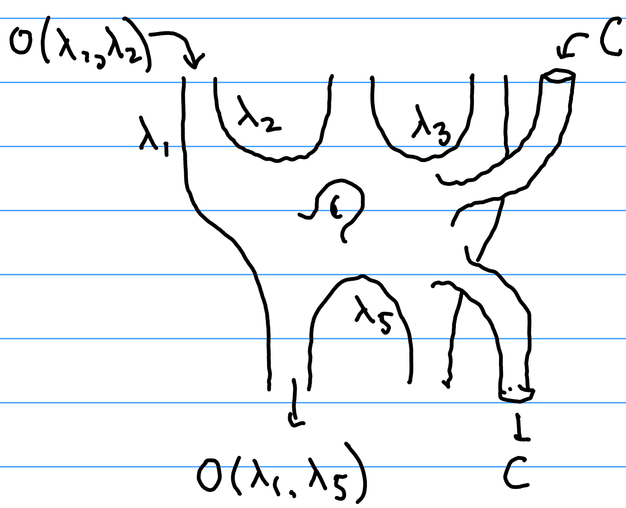
$$C_*(\mathcal{M}) \longrightarrow \mathrm{Hom}\left(\mathbb{C}^{\otimes P_{int}^{in}} \otimes \bigotimes_{p \in P_\partial^{in}} \mathcal{O}(p),\; \mathbb{C}^{\otimes P_{int}^{out}} \otimes \bigotimes_{p \in P_\partial^{out}} \mathcal{O}(p) \right)$$

singular chains

where $\mathcal{O}(p) := \mathcal{O}(\lambda, \lambda')$

D-brane to left of p

D-brane to right of p

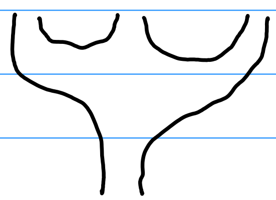


(think: 'propagating strings').

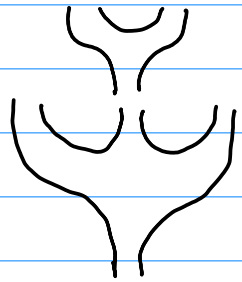
- These maps are required to satisfy relations coming from gluing surfaces at boundary and interior points.

Lem: There's an A_∞ category with

- objects = branes
- $\hom(X, Y) := \mathcal{O}(X, Y)$
- m_k is the map corresponding to the fundamental cycle of the moduli space of discs with
  - no interior punctures
  - k incoming boundary punctures
  - 1 outgoing boundary puncture

 $= m_3$

Pf: The boundary of the fundamental cycle is equal to a sum of glued discs

 $\longleftrightarrow m_3(-, m_2(-,-), -)$

corresponding to terms in the A_∞ rel'ns.

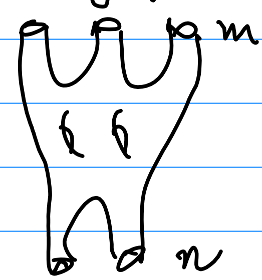
This defines the map OCTCFT $\longrightarrow$ A_∞ cat.

There's another forgetful map:

$$\mathrm{OCTCFT} \xrightarrow{} \mathrm{TCFT}$$

forget branes

TCFT $=$ vector space C with maps $C^{\otimes m} \to C^{\otimes n}$ coming from



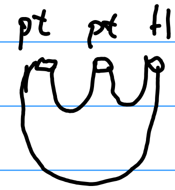
The resulting iso of TCFT's is called 'closed-string MS', and was the first version to be studied; the introduction of branes came later.

The TCFT's arising from $A(X, \omega)$, $B(X, J)$ both have $C = H^*(X)$. So they give rise to operations

$$H^*(X)^{\otimes m} \longrightarrow H^*(X)^{\otimes n}$$

For $A(X, \omega)$, these operations are called <u>Gromov-Witten invariants</u>. They count holomorphic maps from a Riemann surface into X.

E.g. # (lines through 2 points and a hyperplane in $\mathbb{CP}^2$) $= 1$



$\Rightarrow$ the map

$$H_*(\mathcal{M}_{0,3}) \longrightarrow \mathrm{Hom}\left(H^*(\mathbb{CP}^2)^{\otimes 3}, \mathbb{C}\right)$$

sends $[pt] \longmapsto \left(PD(pt) \otimes PD(pt) \otimes PD(H) \mapsto 1\right)$.

For $B(X, J)$, these operations come from Hodge theory. (at least for genus 0).

Therefore, MS predicts an isomorphism

$$H^*(X) \simeq H^*(X^\vee)$$

intertwining GW invts with Hodge theoretic invariants. This can be used to predict GW invts: e.g.

d	# deg-d, $g=0$ curves on a smooth deg-5 hypersurface in $\mathbb{CP}^4$
1	2875
2	609250
3	317 206 375
	$\vdots$