HMS for the affine line and the projective line

$Y = \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[z])$

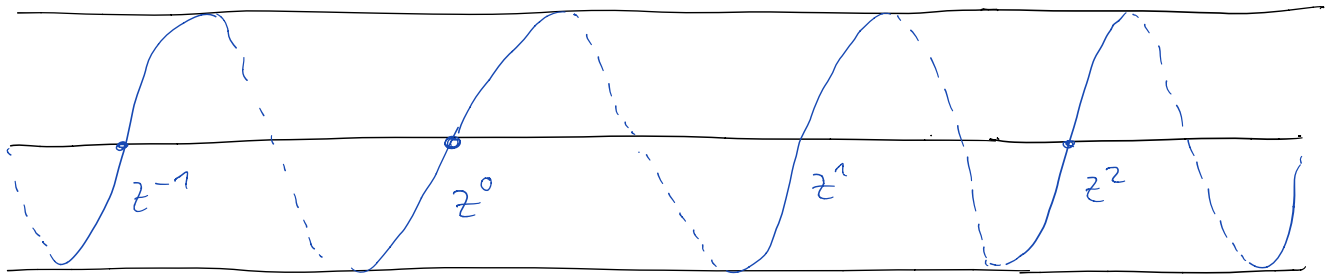
B-side: Consider $\mathcal{O}$, $\mathcal{O}_a$ $a \neq 0$, $\mathcal{O}_0$

Recall:

$\mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \mathbb{C}[z]$ $\quad$ $\mathrm{Ext}^1(\mathcal{O}_a, \mathcal{O}) = \mathbb{C}$

$\mathrm{Hom}(\mathcal{O}, \mathcal{O}_a) = \mathbb{C}$ $\quad$ $\mathrm{Hom}^*(\mathcal{O}_a, \mathcal{O}_a) = \mathbb{C}[\vartheta]/\vartheta^2$

Last talk: Mirror to $\mathbb{A}^1 - \{0\}$.



Explain that we want to get rid of the negative powers. → want to stop it from wrapping in the negative infinite direction

Def A 2-dim symplectic manifold with stops is a pair (X, σ)

X: 2-dim exact symplectic manifold with boundary.

$\sigma \subset \partial X$: finite collection of points called stops

Important: We still think of the boundary as "infinity"

Def $Fuk(X, \sigma)$

objects: $(L, \partial L) \subset (X, \partial X - \partial)$

$hom^*(K, L) := \mathbb{C}\langle L_0 \cap w(L_1) \rangle$

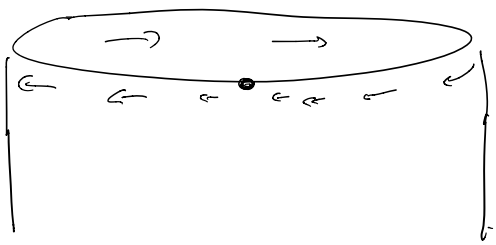
apply w no matter if $L_0 \pitchfork L_1$ or not.

where $w : (X, \partial X) \longrightarrow (X, \partial X)$ wraps near the boundary/infinity as far as possible without going passed the steps

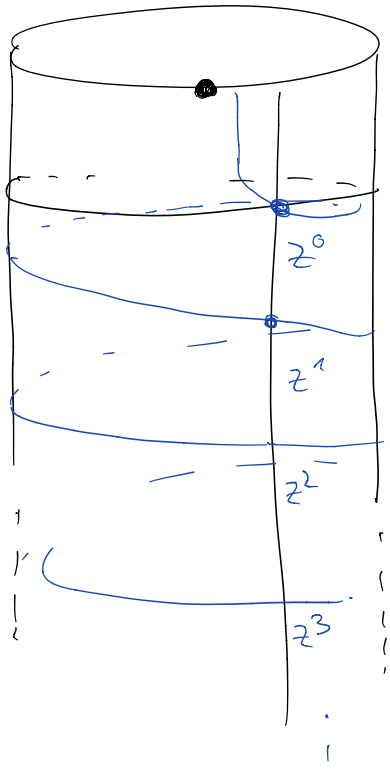
We will see this in action in just a moment..

The rest is like the definition of the wrapped Fukaya category in Augustinas's talk.

w at the boundary:



A-side mirror to $\mathbb{A}^1$:

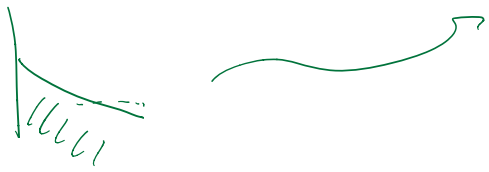


$(S^1, \xi_a) \longleftrightarrow \mathcal{O}_a$

$\Rightarrow \mathrm{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{C}\langle \mathbb{R} \cap w(\mathbb{R}) \rangle = \mathbb{C}[z]$

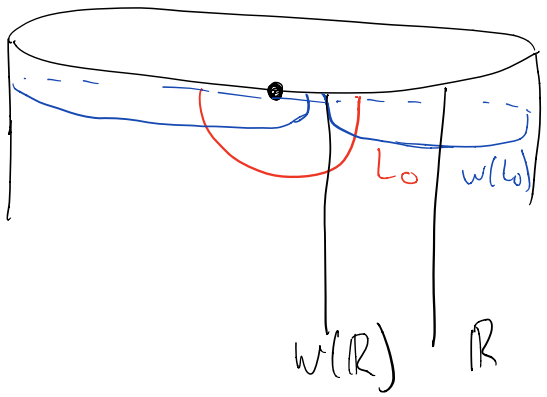
(There are no disks between $\mathbb{R}$ and $w(\mathbb{R})$ with finite energy.

might think



is a disk.

What about mirror to $\mathcal{O}_0$?



New Lagrangian.

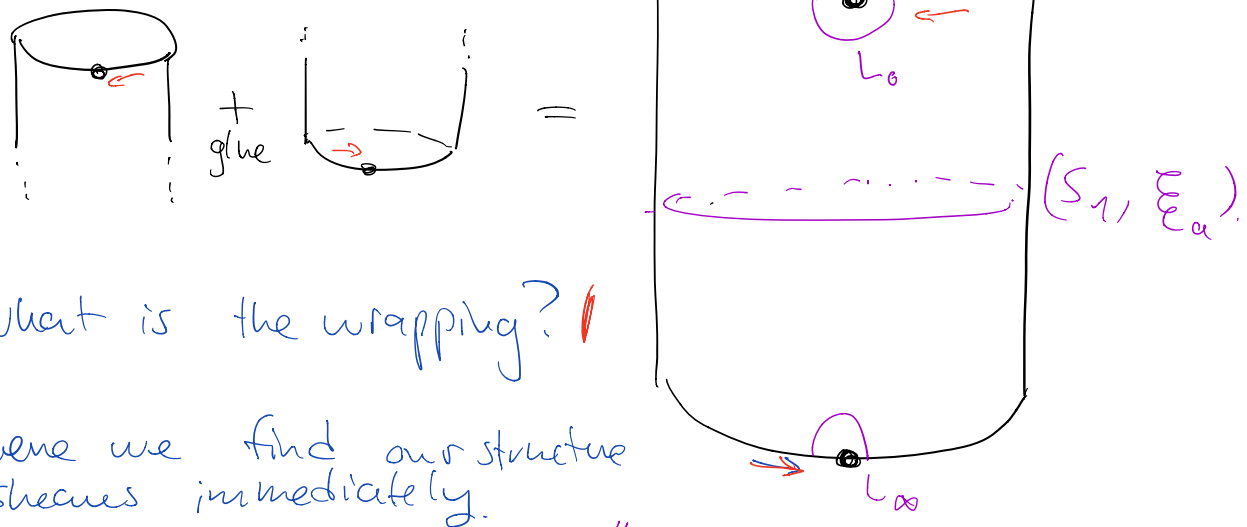
$\Rightarrow \mathrm{Hom}(L_0, \mathbb{R}) = \mathbb{C}\langle L_0 \cap w(\mathbb{R}) \rangle = \mathbb{C}$

$\Rightarrow \mathrm{Hom}(L_0, L_0) = \mathbb{C}\langle L_0 \cap w(L_0) \rangle = \mathbb{C}^{\oplus 2}$

$\mathrm{Hom}(L_0, (S^1, \xi_a)) = 0$. ($L_0$ lives near infinity.)

$Y = \mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$

$\mathbb{A}^1 - \{0\}$



What is the wrapping?

Here we find our structure sheaves immediately.

$\mathcal{O}_a \quad a \neq 0, \quad \mathcal{O}_0, \mathcal{O}_\infty$

But the structure sheaves don't generate the derived category of coherent sheaves!

Need Mirrors $L(k)$ to $\mathcal{O}(k)$ $k \in \mathbb{Z}$.

Hom's of $\mathcal{O}(k)$: (from Matt's talk)

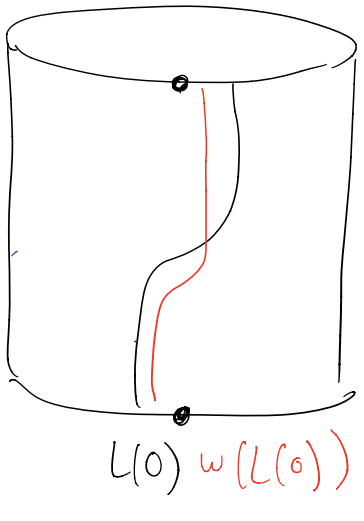
homogenous coordinates: x, y

$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) =$

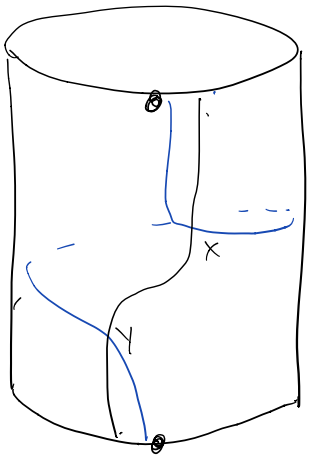
$$\begin{cases} \mathbb{C}[x,y]_{j-i} \cong \bigoplus_{r+s=j-i} \mathbb{C}\, x^r y^s \cong \mathbb{C}^{j-i+1} & \text{if } i-1 < j \\ 0 & \text{if } i-1 \geq j \end{cases}$$

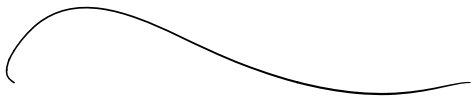
$$\mathrm{Ext}^1(\mathcal{O}(i), \mathcal{O}(j)) = \begin{cases} 0 & \text{if } i-1 \leq j \\ \frac{1}{xy}\mathbb{C}[x^{\pm 1}, y^{\pm 1}]_{j-i+1} \cong \mathbb{C}^{i-j-1} & \text{if } i-1 > j \end{cases}$$

Mirror to $\mathcal{O} = \mathcal{O}(0)$



Mirror to $\mathcal{O}(1)$:





How do compositions work?

B-side: $i \le j \le k$. From description of Homs. these are honest Hom grps

$$\mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(j)) \otimes \mathrm{Hom}(\mathcal{O}(j),\mathcal{O}(k)) \to \mathrm{Hom}(\mathcal{O}(i),\mathcal{O}(k))$$
$$(\cdot x^r y^s) \otimes (\cdot x^{r'} y^{s'}) \longmapsto (\cdot x^{r+r'} y^{s+s'})$$
$$r+s = j-i \qquad r'+s' = k-j$$

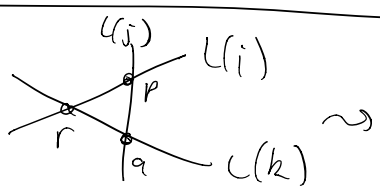
$$\mathrm{Ext}^1(\mathcal{O}(i),\mathcal{O}(j)) \otimes \mathrm{Hom}(\mathcal{O}(j),\mathcal{O}(k)) \to \mathrm{Ext}^1(\mathcal{O}(i),\mathcal{O}(k))$$

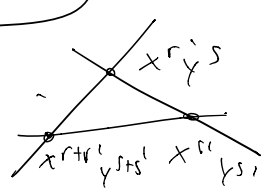
? ? ? ? ?

A-side: $\hom(L(i),L(j)) \otimes \hom(L(j),L(k)) \to \hom(L(i),L(k))$

$$p \otimes q \longmapsto \sum_{r \in \hom(L(i),L(j))} \eta_{pqr}\, r$$

# of disks like

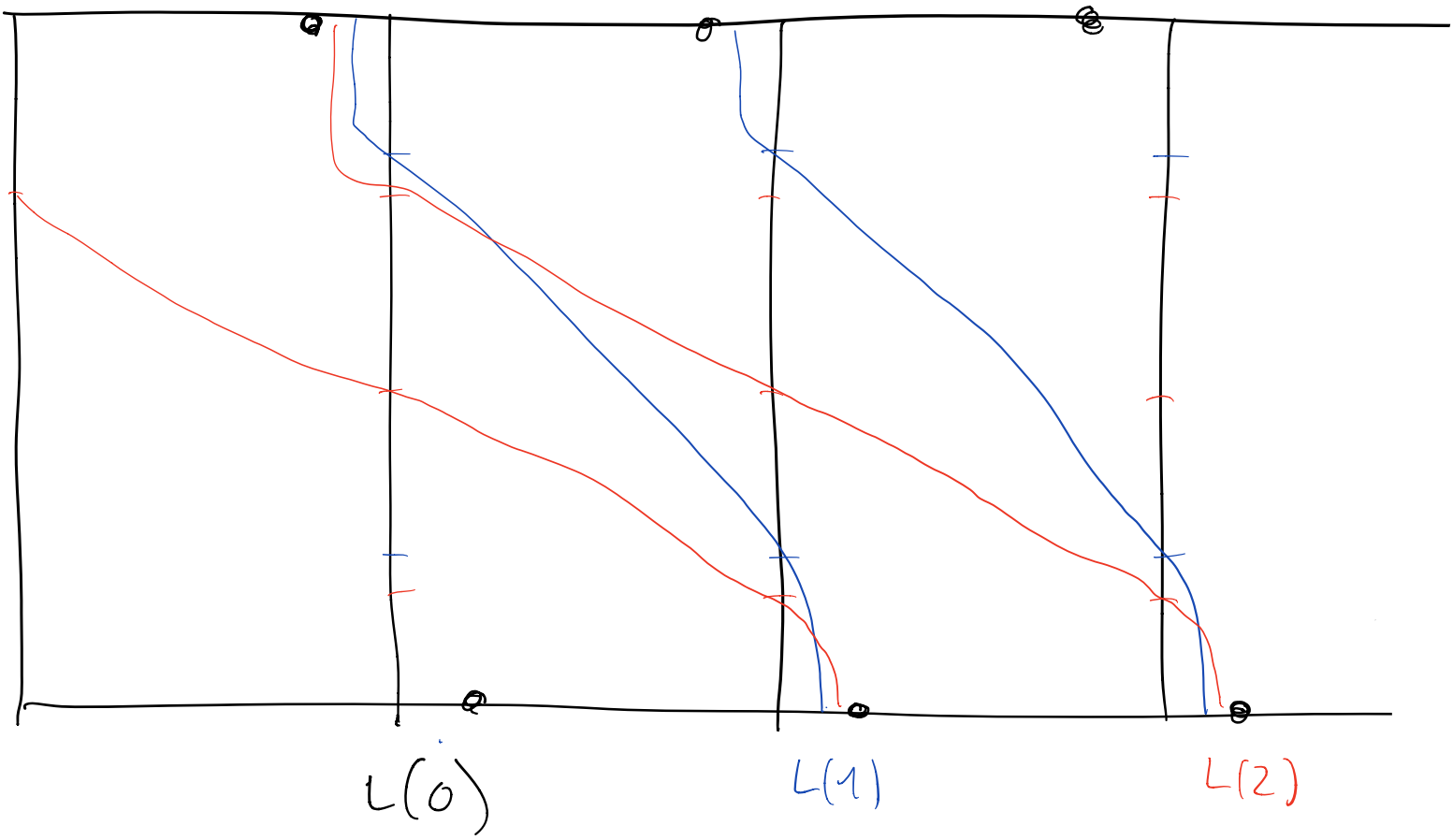




Verify in an example.

Draw universal cover

$$Hom(L(0), L(1)) \otimes Hom(L(1), L(2)) \to Hom(L(0), L(2))$$



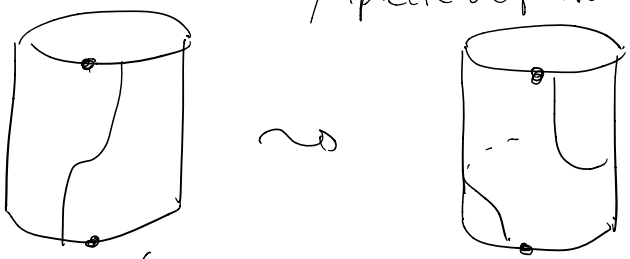
Fact 1: $D^b(Coh(\mathbb{P}^1)) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$

Fact 2: $Fuk(\text{cylinder}, \text{stops}) = \langle L(0), L(1) \rangle$

$\Rightarrow$ HMS

Mirrors of auto equivalences

$$\mathcal{O}(1)\otimes - \;:\; D^b(\mathrm{coh}(\mathbb{P}^1))\circlearrowleft$$
$$\updownarrow$$
φ : (cylinder, steps) $\circlearrowleft$ by "twisting" / "jumping over step"
symplectomorphism.



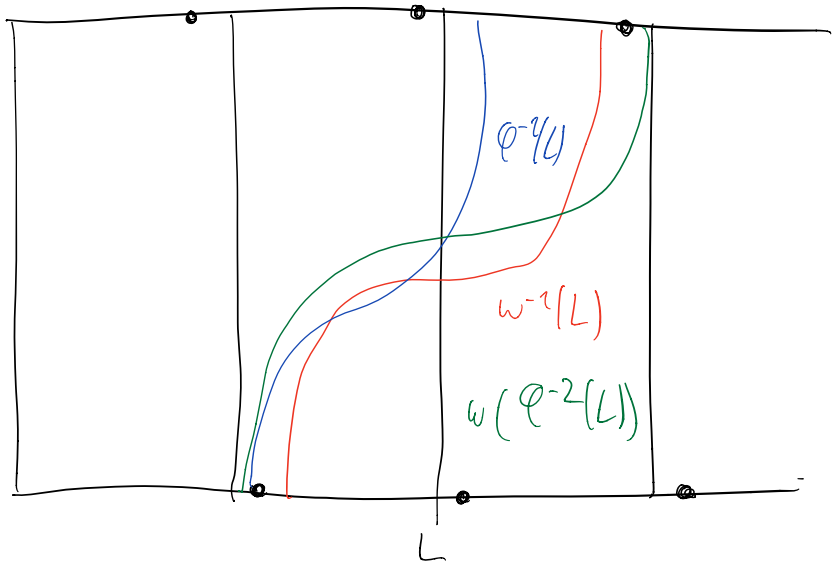
$\rightsquigarrow$ $\varphi_* : \mathrm{Fuk}(\text{cylinder, steps})\circlearrowleft$

$\Rightarrow$ •) $\varphi_*(L(i)) = L(i+1)$

•) $\mathrm{Hom}(\varphi_*(L(i)), \varphi_*(L(j))) = \mathrm{Hom}(L(i), L(j))$

Mirror of Serre duality

$$\begin{aligned}\hom(L(i),L(j)) &= \mathbb{C}\langle L(i)\cap w(L(j))\rangle \\ &= \mathbb{C}\langle L(j)\cap \tilde{w}^{-1}(L(i))\rangle \\ &= \mathbb{C}\langle L(j)\cap w(\varphi^{-2}(L(i)))\rangle\end{aligned}$$



twist once

get closer to one step to the left and right

ε

get closer to one step to the left and right

twist once.

twist once

$$\mathrm{Ext}^0(\mathcal{O}(i),\mathcal{O}(j)) \cong \mathrm{Ext}^1(\mathcal{O}(j),\mathcal{O}(-2)\otimes\mathcal{O}(i))^\vee$$