HOMOLOGICAL MIRROR SYMMETRY FOR THE ELLIPTIC CURVE

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1. Introduction

The original form of mirror symmetry was supposed to be a duality between compact Calabi-Yau manifolds. Since these are kind of complicated, so far we have focused on easier non-compact examples. But the case of 1-dimensional Calabi-Yaus (=elliptic curves) is not too complicated, so we can still see mirror symmetry fairly easily for them.

On the A side, we have the symplectic torus

$$(X = \mathbb{R}^2/\mathbb{Z}^2, \omega = dx \wedge dy).$$

On the B side, we should have an elliptic curve defined over the field of definition of the Fukaya category of (X, ω). Unlike in previous examples, we should no longer take this to be $\mathbb{C}$: instead everything should be defined over the *Novikov field*

$$\Lambda = \mathbb{C}((T^{\mathbb{R}})) = \left\{ \sum_{k \geq 0} a_k T^{b_k} \mid a_k \in \mathbb{C}, b_k \in \mathbb{R}, b_0 < b_1 < \cdots, b_k \to \infty \right\}.$$

The mirror to X is the elliptic curve

$$X^\vee = \Lambda^\times / T^{\mathbb{Z}}.$$

This is called the *Tate curve* over Λ. It can be endowed canonically with the structure of an algebraic variety over Λ in an analogous way to the curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ for $0 < |q| < 1$.

2. B side

On the B side, we have the Tate curve over the Novikov field. To motivate this construction, recall that any elliptic curve over $\mathbb{C}$ can be written analytically as

$$\begin{aligned} \frac{\mathbb{C}}{\mathbb{Z} + \mathbb{Z}\tau} &\xrightarrow{\sim} \mathbb{C}^\times/q^{\mathbb{Z}} \\ w &\longmapsto e^{2\pi i z} \end{aligned}$$

where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$ and $q = e^{2\pi i \tau}$ satisfies $0 < |q| < 1$. Since $\mathbb{C}^\times/q^{\mathbb{Z}}$ is a compact Riemann surface, it can be given the structure of an algebraic variety in a unique way.

A similar theory works for a class of fields called "complete non-Archimedean normed fields", of which the Novikov field Λ is an example. The Tate curve over Λ is

$$X^\vee = \Lambda^\times / T^{\mathbb{Z}}.$$

This can be interpreted as an "analytic space over Λ", which can again be uniquely realised as an algebraic variety. It is a smooth elliptic curve over Λ.

Remark 1. Another way to get the Tate curve is to take a family of elliptic curves over $\operatorname{Spec} \mathbb{C}[[T]]$ degenerating from a smooth curve at the generic fibre to a nodal cubic over $T = 0$, and base change along $\operatorname{Spec} \Lambda \to \operatorname{Spec} \mathbb{C}[[T]]$. So the appearance of the Tate curve rather than a smooth elliptic curve over $\mathbb{C}$ is an incarnation of the general principle that mirror symmetry for compact Calabi-Yaus works "at the boundary of the moduli space".

Coherent sheaves on $X^\vee$?

- The structure sheaf $\mathcal{O}_{X^\vee}$.
- The skyscraper sheaves $\mathcal{O}_q$ for $q \in X^\vee = \Lambda^\times/T^{\mathbb{Z}}$.
- The line bundle
$$\mathcal{L} = \frac{\Lambda^\times \times \Lambda}{(z, v) \sim (Tz, T^{-1/2} z^{-1} v)} \longrightarrow \frac{\Lambda^\times}{T^{\mathbb{Z}}} = X^\vee.$$
- More generally, given any "multiplier" $\phi \colon \Lambda^\times \to GL_n(\Lambda)$ (given by convergent Laurent series), there is a rank n vector bundle
$$\mathcal{V}_\phi = \frac{\Lambda^\times \times \Lambda^n}{(z, v) \sim (Tz, \phi(z) v)}$$
associated to ϕ. It turns out that all vector bundles on $X^\vee$ occur in this way.

Morphisms?

- $\mathrm{Hom}(\mathcal{O}, \mathcal{O}_q) = \Lambda$.
- $\mathrm{Hom}(\mathcal{L}, \mathcal{O}_q)$ is 1-dimensional. If we fix a preimage z of q in $\Lambda^\times$, then there is a well-defined basis element $\mathrm{ev}_z \colon \mathcal{L} \to \mathcal{O}_q$. It satisfies $\mathrm{ev}_{Tz} = T^{-1/2} z^{-1} \mathrm{ev}_z$.
- A morphism $\mathcal{O} \to \mathcal{L}$ is the same thing as a global section of $\mathcal{L}$, which is the same thing as an analytic map
$$\theta \colon \Lambda^\times \longrightarrow \Lambda$$
satisfying $\theta(Tz) = T^{-1/2} z^{-1} \theta(z)$. Since θ is analytic, it can be expanded as a Laurent series
$$\theta(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$
where $a_k \in \Lambda$. The quasi-periodicity gives the recurrence relation $a_{k+1} = T^{k+\frac{1}{2}} a_k$, so up to scale there is a unique solution
$$\theta(z) = \sum_{k=-\infty}^{\infty} T^{\frac{1}{2}k^2} z^k,$$
(which does indeed converge for $z \in \Lambda^\times$.)

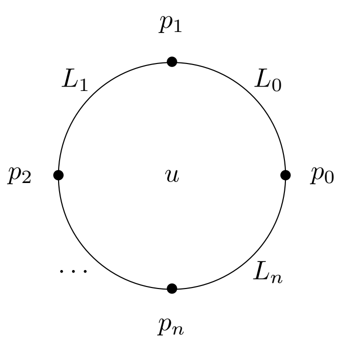
Compositions? We've set it up so that
$$m_2(\theta, \mathrm{ev}_z) = \mathrm{ev}_z \circ \theta = \theta(z).$$

3. *A* side

We briefly recall the Fukaya category $\mathrm{Fuk}(X, \omega)$. The objects are pairs (L, ξ), where $L \subseteq X$ is a compact Lagrangian submanifold, and ξ is a U_1-local system on L. (The Lagrangian L should also be equipped with extra structures called a *grading* and a *spin structure*, which we'll follow the convention of brushing under the rug.) If L_0 and L_1 are transverse Lagrangians, then the hom spaces are given by
$$\hom((L_0, \xi_0), (L_1, \xi_1)) = \bigoplus_{p \in L_0 \cap L_1} \mathrm{Hom}(\xi_{0,p}, \xi_{1,p}) p.$$
This space is equipped with a grading coming from the gradings on L_0 and L_1.

Remark 2. In previous examples, we had an exact symplectic manifold (or Landau-Ginzburg model), so we were able to restrict to exact Lagrangians and work with vector spaces and local systems over $\mathbb{C}$. In this case, X is *not* exact, so we have to work with vector spaces over Λ. So in this context, "U_1-local system" means a bundle of 1-dimensional Λ-vector spaces with connection (or locally constant transition functions), such that the monodromy around a loop lies in
$$U_1(\Lambda) = \{a_0 T^0 + a_1 T^{b_1} + \cdots \in \Lambda \mid a_0 \neq 0, b_k > 0 \text{ for } k > 0\}.$$
So in particular, $\mathrm{Hom}(\xi_{0,p}, \xi_{1,p})$ is a 1-dimensional Λ-vector space, so $\hom((L_0, \xi_0), (L_1, \xi_1))$ is a Λ-vector space of dimension $|L_0 \cap L_1|$.

The A_∞ operations
$$m_n \colon \hom((L_0, \xi_0), (L_1, \xi_1)) \otimes \cdots \otimes \hom((L_{n-1}, \xi_{n-1}), (L_n, \xi_n)) \longrightarrow \hom((L_0, \xi_0), (L_n, \xi_n))$$
are given by
$$m_n(f_1 p_1, \ldots, f_n p_n) = \sum_{\substack{p_0 \in L_0 \cap L_n \\ u \in M_0(p_0, \ldots, p_n)}} \pm T^{\omega(u)} \mathrm{mon}_{\partial u}(\xi) p_0,$$
where $M_0(p_0, \ldots, p_n)$ is the space of pseudo-holomorphic disks



and $\mathrm{mon}(\xi)\colon \xi_{0,p_0} \to \xi_{n,p_0}$ is the "monodromy map"

$$\xi_{0,p_0} \cong \xi_{0,p_1} \xrightarrow{f_1} \xi_{1,p_1} \cong \cdots \cong \xi_{n-1,p_n} \xrightarrow{f_n} \xi_{n,p_n} \cong \xi_{n,p_0},$$

where the isomorphisms are given by monodromy along the intervals in the boundary of the disk.

In our example $X = \mathbb{R}^2/\mathbb{Z}^2$, $\omega = dx \wedge dy$. All compact Lagrangians are Hamiltonian isotopic to a closed geodesic, i.e. to the image of a straight line of rational slope in $\mathbb{R}^2$. In this context, a grading on a straight line Lagrangian L is equivalent to a choice of phase $\alpha \in \mathbb{R}$ so that $e^{i\pi\alpha}$ is parallel to L. The degree of a point $p \in L_1 \cap L_2$ is then $\lfloor \alpha_1 - \alpha_2 \rfloor$. In all our examples, we will choose phases in $[-\pi/2, \pi/2)$.

Here are some examples:

- $L_0 = \{y = 0\}$, trivial local system.
- $L_1 = \{y = -x\}$, trivial local system.
- For $p = (a, b) \in \mathbb{R} \times U_1(\Lambda)$, set $L_p = \{x = a\}$, local system with monodromy b.

What are the hom spaces?

- $L_0 \cap L_1 = \{s = (0, 0)\}$. We can canonically identify the two fibres of the trivial local system, so
$$\hom(L_0, L_1) = \Lambda s.$$
Since L_0 has phase 0 and L_1 has phase $-1/4$, s has degree 0.
- $L_0 \cap L_p = \{e_p = (a, 0)\}$. We will fix a trivialisation of the local system ξ_p at e_p, so that we can identify
$$\hom(L_0, L_p) = \Lambda e_p.$$
Again, e_p has degree 0.
- $L_1 \cap L_p = \{f_p = (a, -a)\}$. We will parallel transport the trivialisation of ξ_p at $e_p = (a, 0)$ along L_p to $(a, -a)$ to give a trivialisation at f_p, and hence
$$\hom(L_1, L_p) = \Lambda f_p.$$
Again f_p has degree 0.

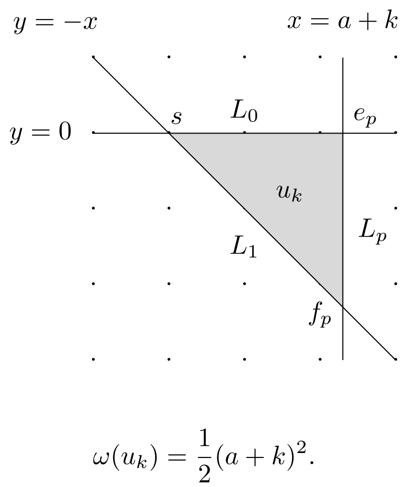
Remark 3. Note that as Lagrangians, $L_{(a,b)} = L_{(a+k,b)}$ for $k \in \mathbb{Z}$. But because we used $a \in \mathbb{R}$ to define the trivialisation of ξ_p at f_p, we do not have $f_{(a,b)} = f_{(a+k,b)}$ as morphisms $L_1 \to L_p$. In fact, we have $f_{(a+k,b)} = b^{-k} f_{(a,b)}$.

For degree reasons, $m_1 = 0$. What about

$$m_2\colon \hom(L_0, L_1) \otimes \hom(L_1, L_p) \longrightarrow \hom(L_0, L_p)?$$

$$m_2(s, f_p) = \sum_{k=-\infty}^{\infty} T^{\omega(u_k)} \mathrm{mon}_{\partial u_k}(\xi) e_p,$$

where u_k is the holomorphic disk



The symplectic area of u_k is

$$\omega(u_k) = \frac{1}{2}(a + k)^2.$$

To compute the monodromy, remember that we chose the identifications of the fibres of the local systems so that the monodromy is 1 when $k = 0$. In general, we pass an extra k times around the loop L_p, so we get

$$\mathrm{mon}_{\partial u_k}(\xi) = b^k.$$

So

$$m_2(s, f_p) = \sum_{k=-\infty}^{\infty} T^{\frac{1}{2}(a+k)^2} b^k e_p = T^{\frac{1}{2}a^2} \sum_{k=-\infty}^{\infty} T^{\frac{1}{2}k^2} (T^a b)^k e_p.$$

4. Mirror symmetry

How do all our examples match up? Fix $(a, b) \in \mathbb{R} \times U_1(\Lambda)$ and set $p = (a, b)$, $z = T^a b$ and $q = zT^{\mathbb{Z}} \in X^\vee$. Then:

$$\begin{aligned} \mathrm{Fuk}(X, \omega) &\longleftrightarrow D^b\mathrm{Coh}(X^\vee) \\ L_0 &\longleftrightarrow \mathcal{O} \\ L_1 &\longleftrightarrow \mathcal{L} \\ L_p &\longleftrightarrow \mathcal{O}_q \\ (s\colon L_0 \to L_1) &\longleftrightarrow (\theta\colon \mathcal{O} \to \mathcal{L}) \\ (e_p\colon L_0 \to L_p) &\longleftrightarrow (1\colon \mathcal{O} \to \mathcal{O}_q) \\ (f_p\colon L_1 \to L_p) &\longleftrightarrow (T^{\frac{1}{2}a^2}\mathrm{ev}_z\colon \mathcal{L} \to \mathcal{O}_q). \end{aligned}$$

Note that we're forced to put the weird factor $T^{\frac{1}{2}a^2}$ in the last correspondence in order to make the compositions work. But we could also have guessed that it was needed because f_p and ev_z have different periodicity as we add integers to $a \in \mathbb{R}$: $f_{(a+1,b)} = b^{-1}f_{(a,b)}$, but $\mathrm{ev}_{Tz} = T^{-\frac{1}{2}}z^{-1}\mathrm{ev}_z = T^{a-\frac{1}{2}}b^{-1}\mathrm{ev}_z$.

Other examples?

- On the B side, we can also take powers $\mathcal{L}^{\otimes n}$ of the line bundle $\mathcal{L}$. Sections are now functions $f(z)$ satisfying $f(Tz) = T^{-\frac{n}{2}}z^{-n}f(z)$. The recurrence relation on the Laurent coefficients now relates a_k to a_{k+n}, so there are n linearly independent solutions $\theta_0(z), \ldots, \theta_{n-1}(z)$. On the A side, this corresponds to the line $L_n = \{y = -nx\}$. The n theta functions correspond to the n intersection points
$$L_0 \cap L_n = \{s_j = (j/n, -j) \mid j = 0, 1, \ldots, n-1\}.$$
The computation of the compositions with the unique intersection point in $L_n \cap L_p$ in the Fukaya category is very similar to the case $n = 1$.
- On the A side, we can also take Lagrangians of arbitrary rational slope m/n for m, n coprime. These correspond to stable(=indecomposable) vector bundles on $X^\vee$ of rank n and degree $-m$. Note that these are at least parametrised by the same set: on the A side there is a choice up to $\mathbb{R}/\mathbb{Z} \times U_1(\Lambda) \cong X^\vee$ of the position of the line and monodromy of the local system, and on the B side a theorem of Atiyah says there is a unique stable bundle with each determinant in $\mathrm{Pic}^n(X^\vee) \cong X^\vee$.

There are a lot of interesting classical structures on $D^b\mathrm{Coh}(X^\vee)$ and on particular objects in it. It is quite fun to try translating these across to the Fukaya category $\mathrm{Fuk}(X, \omega)$:

- Since $X^\vee$ is an elliptic curve, it acts on itself by translations, and hence on its derived category. As a group, $X^\vee = \mathbb{R}/\mathbb{Z} \times U_1(\Lambda)$. On the Fukaya category side, the action of $a \in \mathbb{R}/\mathbb{Z}$ is by horizontal translations, and the action of $b \in U_1(\Lambda)$ is by tensoring with the global local system with monodromy b around a vertical loop and trivial monodromy around a horizontal loop.
- Dually, the group $\mathrm{Pic}^0(X^\vee) \cong X^\vee = \mathbb{R}/\mathbb{Z} \times U_1(\Lambda)$ of degree 0 line bundles acts on $D^b\mathrm{Coh}(X^\vee)$ by tensoring. On the Fukaya side, $\mathbb{R}/\mathbb{Z}$ acts now by vertical translations, and $b \in U_1(\Lambda)$ acts by tensoring with the global local system with trivial monodromy around a vertical loop and monodromy b around a horizontal loop.
- On $X^\vee$, for any n-torsion point $x \in X^\vee[n]$, the translation $t_x^*\mathcal{L}^{\otimes n}$ is isomorphic to $\mathcal{L}^{\otimes n}$. (This is true for any degree n line bundle on any elliptic curve.) An important fact in the theory of elliptic curves is that this does *not* extend to an action of $X^\vee[n]$ on the line bundle $\mathcal{L}^{\otimes n}$, i.e. we can't choose isomorphisms compatibly for all torsion points. (The best one can get is an action of a central extension of $X^\vee[n]$ by μ_n called a Heisenberg group.) On the Fukaya side, the n-torsion points are $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \times \mu_n \subseteq \mathbb{R}/\mathbb{Z} \times U_1(\Lambda)$. Since L_n is manifestly invariant under $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-translations we get an action of this group. We also get an action of μ_n using the fact that L_n wraps n times around the torus vertically, so the monodromy of a global vertical μ_n-local system is trivial. But we can't get these to commute, because the trivialisation of the local system on L_n interacts in an interesting way with translations.
- On the elliptic curve $X^\vee$, there is a *Poincaré line bundle* $\mathcal{P}$ on $X^\vee \times X^\vee$, which is degree 0 on every fibre of both projections. (This realises the isomorphism $X^\vee \cong \mathrm{Pic}^0(X^\vee)$.) This induces an autoequivalence of the derived category
$$\begin{aligned} D^b\mathrm{Coh}(X^\vee) &\xrightarrow{\sim} D^b\mathrm{Coh}(X^\vee) \\ \mathcal{F} &\longmapsto R(\mathrm{pr}_2)_*(\mathcal{P} \otimes \mathrm{pr}_1^*\mathcal{F}). \end{aligned}$$
This is quite nontrivial, and (up to shifts) interchanges skyscraper sheaves and degree 0 line bundles. But on the Fukaya category, this is nothing but a $90°$ rotation of the torus X!