HMS for the cylinder

HMS by Examples
Reading Group - Talk 5

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1 Introduction

In this talk/note, we will give a concrete example of homological mirror symmetry (HMS) in action by looking at the example of the cylinder. We closely follow Nick Sheridan's notes [9] written for this reading group, and flesh out some details. Any mistakes in this note are the author's own.

Recall the basic (roughly stated) statement of mirror symmetry: for X a symplectic manifold there is conjectured to be an algebraic variety Y (its mirror pair) which satisfies the derived equivalence

$$D^b(\mathsf{Coh}(Y)) \simeq D^b \, \mathsf{Fuk}(X),$$

where on left hand side we have the bounded derived category of coherent sheaves on Y, and on the right hand side the bounded derived Fukaya category of X.

2 B-side

The aim of this talk is to show that the variety (scheme) that is mirror to the cylinder is $Y := \operatorname{Spec} \mathbb{C}[z^{\pm 1}] = \mathbb{C}^*$. Since we work over the field $\mathbb{C}$, Y is in fact an algebraic torus (see [4]). Denote the structure sheaf of Y by $\mathcal{O}_Y =: \mathcal{O}$ and the skyscraper sheaves at p by $\mathcal{O}_p$.

Remark 2.1. Under the equivalence of categories $\mathsf{QCoh}(\mathbb{C}^*) \simeq \mathsf{Mod}\text{-}\mathbb{C}[z^{\pm 1}]$ [8, §3], the structure sheaf $\mathcal{O}$ corresponds to the ring $\mathbb{C}[z^{\pm 1}]$ and the skyscraper $\mathcal{O}_a$ to the ring $\mathbb{C}[z^{\pm 1}]/(z-a) \cong \mathbb{C}$. The moral of the story here is that $\mathsf{QCoh}(\mathbb{C}^*) \xrightarrow{\sim} \mathsf{Mod}\text{-}\mathbb{C}[z^{\pm 1}]$ means take global sections; we have $\mathcal{O}(\mathbb{C}^*) \cong \mathbb{C}[z^{\pm 1}]$ and $\mathcal{O}_a(\mathbb{C}^*) \cong \mathbb{C}$ because the skyscraper sheaf $\mathcal{O}_a$ has stalk $\mathbb{C}$ when evaluated on opens containing a, and zero stalks otherwise.

Remark 2.2. The derived category $D^b(Y) = D^b(\mathbb{C}^*)$ is generated by the structure sheaves $\mathcal{O}$. Therefore, the proof of HMS for the cylinder only requires us to find mirror objects (and Hom sets, relations, etc.) for $\mathcal{O}$. However, we will also find the mirrors to the skyscrapers so that we can see more of the principles used in HMS in action.

2.1 Computing Hom sets of sheaves

To prove HMS, we must find what the morphisms between our mirror objects look like. So let's compute the morphisms between the objects on the B-side. Here, we use the notation and convention (see [3])

$$\operatorname{Hom}^*_{D(R)}(M, N) \cong \operatorname{Ext}^*_R(M, N).$$

Firstly, as we saw in Matt's talk [3], the Hom between two skyscrapers will be given by

$$\operatorname{Hom}^*(\mathcal{O}_a, \mathcal{O}_b) \cong \begin{cases} 0 & \text{if } a \neq b \\ \mathbb{C}[\theta]/\theta^2 \text{ with } |\theta| = 1 & \text{if } a = b \end{cases}$$

The $a \neq b$ case being zero is geometrically reasonable due to having the stalks of skyscrapers at distinct points, as mentioned in Matt's talk. For the $a = b$ case, to compute this explicitly, one can take a resolution of $R/(z-a)$, take contravariant Hom, and compute the cohomology of the resulting complex as one would normally do for Ext computations. we will do this in more detail for one of the cases below.

Recall also the following result (also mentioned in Matt's talk [3]).

Proposition 2.3 ([5, p. 234])**.** *For $\mathcal{F}$ a coherent sheaf on X a variety, we have*

$$\operatorname{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F})$$

for all $i \geq 0$.

We also quote part of a cohomology vanishing theorem of Serre from [5, p. 215].

Theorem 2.4 (Serre)**.** *Let X be a noetherian scheme. Then X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > 0$.*

For the Hom's involving the structure sheaf, we have

$$\operatorname{Hom}^*(\mathcal{O}, \mathcal{O}) \cong \operatorname{Ext}^*(\mathcal{O}, \mathcal{O}) \cong H^*(Y, \mathcal{O}) = \mathbb{C}[z^{\pm 1}]$$
$$\operatorname{Hom}^*(\mathcal{O}, \mathcal{O}_a) \cong \operatorname{Ext}^*(\mathcal{O}, \mathcal{O}_a) \cong H^*(Y, \mathcal{O}_a) = \mathbb{C}[0]$$
$$\operatorname{Hom}^*(\mathcal{O}_a, \mathcal{O}) \cong \operatorname{Ext}^*(\mathcal{O}_a, \mathcal{O}) = \mathbb{C}[1]$$

where the square brackets on the second and third lines denote the degree that $\mathbb{C}$ is in. The first line is by the general fact $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$ (see [5] for example), and cohomology vanishes for degrees $* > 0$ by Theorem 2.4 (Y is affine and $\mathcal{O}$ is quasi-coherent). One can see the second line by noting that the skyscraper sheaf is flasque, and flasque sheaves have vanishing cohomology for degree ≥ 1 [5, p. 208]. We have $\mathbb{C}$ in degree zero since the skyscraper $\mathcal{O}_a$ has a stalk only at a.

For the third line, let's actually compute the Ext groups by taking a resolution. First note that under the equivalence of categories in Remark 2.1, we have $\operatorname{Ext}^*_{\mathbb{C}^*}(\mathcal{O}_a, \mathcal{O}) \cong \operatorname{Ext}^*_R(R/(z-a), R)$ where $R = \mathbb{C}[z^{\pm 1}]$. Let's compute a projective resolution of $R/(z-a)$. We have

$$0 \longrightarrow R \xrightarrow{\cdot(z-a)} R \longrightarrow R/(z-a) \longrightarrow 0$$

so our projective resolution is $0 \to R \xrightarrow{\cdot(z-a)} R$. Taking $\operatorname{Hom}_R(-, R)$ (contravariant!), we get

$$\begin{tikzcd}
0 & \operatorname{Hom}_R(R,R) \arrow[l] \arrow[d] & \operatorname{Hom}_R(R,R) \arrow[l, "\operatorname{Hom}(\cdot(z-a),R)"'] \\
0 & R \arrow[l] & R \arrow[l, "\phi"'] \arrow[u]
\end{tikzcd}$$

To find the map ϕ, we use the canonical isomorphism $\mathrm{Hom}_R(R, M) \cong M$ for M an R-module, given by the maps $f \mapsto f(1_R)$ and $m \mapsto r \cdot m$ in each direction. Starting with an element $f(z) \in R$, we get

$$\begin{tikzcd}
h = g \circ (\cdot(z-a)) : q(z) \mapsto f(z) \cdot q(z) \cdot (z-a) \arrow[d, mapsto] & g : p(z) \mapsto f(z) \cdot p(z) \arrow[l, mapsto] \\
1 \cdot f(z) \cdot (z-a) & f(z) \arrow[u, mapsto] \arrow[l, dashed, mapsto, "\phi"]
\end{tikzcd}$$

So our map ϕ is just multiplication by $(z-a)$. To get the ith Ext groups, all that is left to do is to compute the cohomology of the complex $C^\bullet = (0 \leftarrow R \xleftarrow{\phi} R)$. The 0-th cohomology is zero, because ϕ is injective. The first cohomology is

$$\mathrm{Ext}^1(R/(z-a), R) = H^1(C^\bullet) = \frac{\mathbb{C}[z][z^{-1}]}{(z-a)} \cong \mathbb{C}[a][a^{-1}] \cong \mathbb{C}.$$

So we have a copy of $\mathbb{C}$ in degree one, as the third line says.

2.1.1 Compositions of morphisms

The composition of morphisms is given by

$$\begin{array}{ccc}
\mathrm{Hom}^*(\mathcal{O}, \mathcal{O}) \otimes \mathrm{Hom}^*(\mathcal{O}, \mathcal{O}_a) & \to & \mathrm{Hom}^*(\mathcal{O}, \mathcal{O}_a) \\
f(z) \otimes 1 & \mapsto & f(a)
\end{array}$$

One can check the rest of the compositions in a similar manner.

3 A-side

Let $X := \mathbb{R} \times S^1$ be the (infinite) cylinder, and $\omega = \mathrm{d}r \wedge \mathrm{d}\theta = \mathrm{d}(r\mathrm{d}\theta)$ be a 2-form on it.

Definition 3.1. If $\omega = \mathrm{d}\alpha$ for some α a 1-form, then we say (X, ω) is *exact.*

Fact. If $L \subset X$ is Lagrangian (recall for Lagrangian submanifolds, $\omega|_L = 0$ and $\dim L = \frac{1}{2} \dim X$), we have $\mathrm{d}(\alpha|_L) = 0$.

Definition 3.2. If $\alpha|_L = \mathrm{d}h_L$, then we say L is *exact.*

Note that our setting here is exact Lagrangian submanifolds in an exact symplectic manifold [2, p. 6].

3.1 Recollection of the Fukaya category

Let's briefly recall the basics of the Fukaya category outlined in the second and third talks by Nick [9]. If (X, ω) is a symplectic manifold, then $\mathsf{Fuk}(X, \omega)$ is an A_∞-category with

- **Objects:** Lagrangian submanifolds $L \subset X$ with extra conditions and data (a "brane" structure)
- **Morphisms:**

$$\hom(L_0, L_1) = K\langle L_0 \cap L_1 \rangle = \bigoplus_{p \in L_0 \cap L_1} K \cdot p$$

  with K a field
- **Product:** The compositions of these morphisms are given as follows:

$$m_k : \hom(L_0, L_1) \otimes \hom(L_1, L_2) \otimes \cdots \otimes \hom(L_{k-1}, L_k) \to \hom(L_0, L_k)$$

$$m_k(p_1, \ldots, p_k) := \sum_{p_0, u \in \mathcal{M}_0(p_0, \ldots, p_k)} e^{-2\pi\omega(u)} \cdot p_0$$

  where

$$\mathcal{M}(p_0, \ldots, p_k) := \{u : \Sigma \to X \text{ holomorphic}\}/\sim$$

  where Σ is the disc with $k+1$ marked boundary points z_i, and $u(z_i) = p_i$, $u(z) \in L_i$ for $z \in \partial\Sigma$ between z_i and z_{i+1}. In the sum for the composition m_k, the notation $\mathcal{M}_0$ means the 0-dimensional component of $\mathcal{M}$.

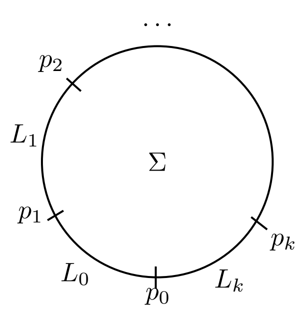

Figure 1: Holomorphic disc Σ with marked boundary points.

But due to the problem of $\mathcal{M}_0$ possibly being infinite, we had to use Gromov compactness and introduce the Novikov field to remedy this. More precisely, we defined the Novikov field

Definition 3.3 (Novikov field)**.**

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i \cdot T^{\lambda_i} \mid a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lim_{i \to \infty} \lambda_i = +\infty \right\}$$

to serve as the coefficient field of $\mathsf{Fuk}(X, \omega)$ instead of $\mathbb{C}$, and we set $\hom(L_0, L_1) = \Lambda\langle L_0 \cap L_1 \rangle$ with

$$m_k(p_1, \cdots, p_k) := \sum_{p_0, u \in \mathcal{M}_0(p_0, \ldots, p_k)} T^{\omega(u)} \cdot p_0$$

3.2 The exact Fukaya category

Definition 3.4. The *exact Fukaya category* of X is like the Fukaya category of X, except for these differences:

- The objects are **exact** Lagrangians.
- $\hom(L_0, L_1) = \mathbb{C}\langle L_0 \cap L_1 \rangle$ (note here that $\mathbb{C}$ is **not** a Novikov field).
- The compositions of morphisms are as before, except here we set $T = 1$.

One may ask why we can do this (put $\mathbb{C}$ as the field and set $T = 1$)? The Novikov field 3.3 was introduced to accommodate the infinite situation (infinite sums), and it worked by Gromov compactness. Note that $\Lambda \to \mathbb{C}$, $T \mapsto 1$ is not well-defined.

The answer as to why we can work with this special Fukaya category is the exactness of our Lagrangian submanifolds (start of §3). Given $p \in L_0 \cap L_1$ and $h_{L_i} \in C^\infty(L_i, \mathbb{R})$, define $A(p) := h_{L_0}(p) - h_{L_1}(p)$.

Lemma 3.5. *If $u \in \mathcal{M}(p_0, \ldots, p_k)$, then*

$$\omega(u) = A(p_0) - \sum_{i=1}^{k} A(p_i).$$

Proof. Recall that $u : \Sigma \to X$, so the pullback u^* brings ω to the setting of the marked disc. Now, use Stokes' theorem, which in general says that

$$\int_{\partial X} \omega = \int_X \mathrm{d}\omega$$

and Definitions 3.1 and 3.2 to deduce the following equalities:

$$
\begin{aligned}
\omega(u) &:= \int_{\Sigma} u^*\omega \\
&= \int_{\Sigma} u^*(\mathrm{d}\alpha) \\
&= \int_{\Sigma} \mathrm{d}(u^*\alpha) \\
&= \int_{\partial\Sigma} u^*\alpha \\
&= \int_{L_0} u^*\alpha|_{L_0} + \cdots + \int_{L_k} u^*\alpha|_{L_k} \\
&= \int_{L_0} u^*\mathrm{d}h_{L_0} + \cdots + \int_{L_k} u^*\mathrm{d}h_{L_k} \\
&= \int_{L_0} \mathrm{d}(u^*h_{L_0}) + \cdots + \int_{L_k} \mathrm{d}(u^*h_{L_k}) \\
&= \int_{\partial L_0} u^*h_{L_0} + \cdots + \int_{\partial L_k} u^*h_{L_k} \\
&= h_{L_0}(p_1) - h_{L_0}(p_0) + h_{L_1}(p_2) - h_{L_1}(p_1) + \cdots + h_{L_k}(p_0) - h_{L_k}(p_k) \\
&= A(p_0) - \sum_{i=1}^{k} A(p_i)
\end{aligned}
$$

where in the last equality, we rearrange the summands to get the $A(p_i)$'s. This is good to see in practice on the disc with three marked points. $\square$

The upshot is that in the exact case, $\omega(u)$ is independent of u, so only one power of T appears in the definition of m_k, so we can set $T = 1$. We now know that the exact Fukaya category $\mathsf{Fuk}^{\mathrm{ex}}(X)$ is $\mathbb{C}$-linear (i.e. the morphism sets of $\mathsf{Fuk}^{\mathrm{ex}}(X)$ have a $\mathbb{C}$-module structure and the compositions of morphisms are $\mathbb{C}$-bilinear maps, i.e. enriched over the category $\mathbb{C}$-Mod [7]). This means that we can compare it with $D^b(Y) = D^b(\mathbb{C}^*)$, because $D^b(Y)$ is also a $\mathbb{C}$-linear category.

Remark 3.6. If we weren't in the exact case, then we would have to work with the Novikov field Λ, and take a variety over Λ. Note that Λ is an algebraically closed field of characteristic zero, so it's a nice field to work over.

3.3 The compact Lagrangian case

The only compact exact Lagrangian in X up to Hamiltonian isotopy is the circle S^1 (which we write as the set $\{r = 0\}$ on the cylinder). It's important to note that we really must consider S^1 at 0 on the cylinder; otherwise, it will not be exact.

Proposition 3.7 ([9, Lec. 2, p. 9]). *If $\omega|_{\pi_2(X,L)} = 0$, i.e. if $\omega : \pi_2(X, L) \to \mathbb{R}$ vanishes and L is compact, then*

$$\mathrm{Hom}^*(L, L) \cong H^*(L)$$

where $H^*(L)$ is the cohomology ring of L with its cup product as graded algebras.

In our case, we know that

$$H^*(S^1) = \mathbb{C}[\theta]/\theta^2$$

where $|\theta| = 1$. So $S^1 \subset X$ is mirror to $\mathcal{O}_1$ (The 1 rather than anything else because this is the trivial monodromy case; see the next subsection). But recall that in §3 we had a statement concerning all of the skyscrapers, not just $\mathcal{O}_1$. Solving this problem will be the focus of the next subsection.

3.4 Local systems

We remedy this by introducing a new type of object (L, ξ) of $\mathsf{Fuk}(X)$ where L is a Lagrangian submanifold and ξ is a $\mathbb{C}$-local system (these are flat vector bundles $\xi \to L$ with unitary holonomy over $\mathbb{C}$ [2, p. 23]). We also introduce a modification of m_k (we follow Remark 2.11 of [2] for this). Consider the isomorphisms $\gamma_i : \xi_i|_{p_i} \to \xi_i|_{p_{i+1}}$ and also consider morphisms $\rho_i : \xi_{i-1}|_{p_i} \to \xi_i|_{p_i}$. Then define $\eta_{u,\rho_1,\ldots,\rho_k} := \gamma_k \circ \rho_k \circ \cdots \circ \gamma_1 \circ \rho_1 \circ \gamma_0$, and then define the new compositions as

$$m_k(\rho_1, \ldots, \rho_k) := \sum_{p_0, u \in \mathcal{M}_0(p_0,\ldots,p_k)} T^{\omega(u)} \cdot p_0 \cdot \eta_{u,\rho_1,\ldots,\rho_k}.$$

What this η does is adds a "monodromy contribution" as we go round the boundary of a holomorphic disc.

3.4.1 The case of S^1's on a cylinder

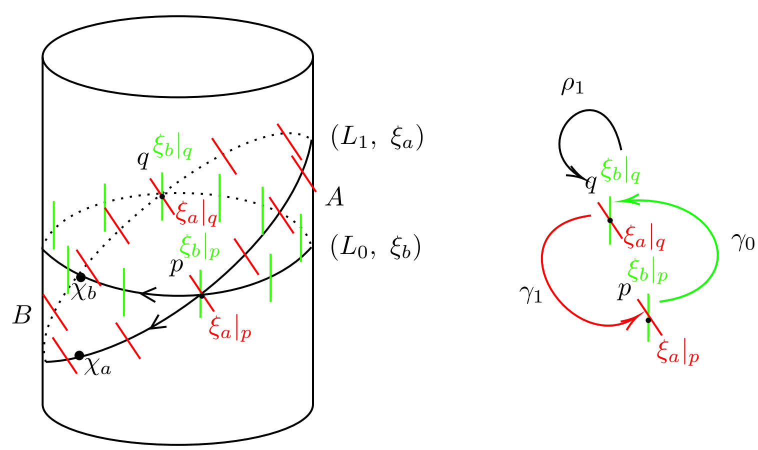

Figure 2: The set-up for the cylinder on the left, and the map η on the right.

In our case, we remove two points denoted χ_a and χ_b on each circle S^1_a and S^1_b. Since $S^1 \setminus \{\mathrm{pt}\}$ is homeomorphic to the open interval $(0,1)$, this allows us to locally trivialise our local systems ξ_a and ξ_b. Then, as we go round the boundaries of the holomorphic discs A and B, the contribution of η is trivial for points in $S^1_*\backslash\chi_*$, however, as we cross the χ_*, there is a monodromy contribution. More precisely, starting from the point p, as we track round B clockwise, we get a contribution of ba^{-1} (because we go round S^1_a opposite to its orientation). On the other hand, as we track round A starting from p, we get trivial monodromy contribution 1 as we have removed no points here. So this gives

$$m_1(p) = q \cdot (1 - ba^{-1})$$

where we take the negative sign on faith (it occurs for reasons that we have not yet discussed). Recall also that here we have set $T = 1$. Finally, doing the same but starting at q gives $m_1(q) = 0$, as in the previous talks.

So now we know that the map

$$m_1 : \hom((S^1,\xi_a),(S^1,\xi_b)) = \mathbb{C}\cdot p \oplus \mathbb{C}\cdot q \to \hom((S^1,\xi_a),(S^1,\xi_b)) = \mathbb{C}\cdot p \oplus \mathbb{C}\cdot q$$

is given by $p \mapsto q \cdot (1 - b^{-1}a)$. If $a \neq b$, then this is an isomorphism so the cohomology is zero (we compute the cohomology because Hom^* is really the cohomology of m_k). If $a = b$ on the other hand, then we have the zero map and for the cohomology we get a two-dimensional vector space. The ring structure comes from the fact that $m_1(q) = 0$. So in summary, we have

$$\mathrm{Hom}^*((S^1,\xi_a),(S^1,\xi_b)) \cong \begin{cases} 0 & \text{if } a \neq b \\ \mathbb{C}[\theta]/\theta^2 \text{ with } |\theta| = 1 & \text{if } a = b \end{cases}$$

which agrees with what we had for skyscrapers $\mathcal{O}_a$ on the A-side, so the object $S^1_a := (S^1,\xi_a)$ is mirror to the skyscraper $\mathcal{O}_a$.

3.5 The wrapped Fukaya category

Now only $\mathcal{O}$ is left to find a mirror to. We follow [1] (and [2]) for this subsection. Note that $\mathrm{Hom}^*(\mathcal{O},\mathcal{O}) = \mathbb{C}[z^{\pm 1}]$ is infinite-dimensional, so $\mathbb{C}\langle L \cap \varphi(L)\rangle$ must too be infinite-dimensional. For this to occur, L must be non-compact. We will require that these non-compact objects should be well behaved at infinity; by this we mean that (in the case of infinite cylinders for example) the Lagrangians will wrap radially round the cylinder as we go to infinity [1]. For simplicity, we define the wrapped Fukaya category with respect to the picture below (following Denis Auroux's talk [1]). For a more comprehensive, precise discussion, see Auroux's notes [2].

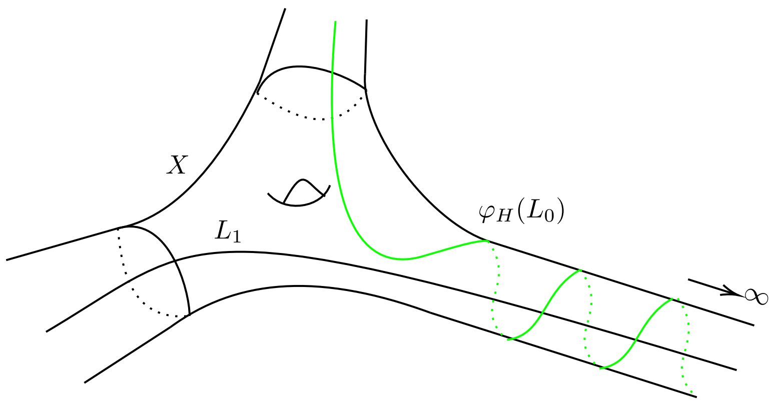

Figure 3: A symplectic manifold X with infinite cylindrical ends. Note the wrapping in the limit of the Lagrangian $\varphi_H(L)$.

Definition 3.8 (Wrapped Fukaya category, [1])**.** Denote the *wrapped Fukaya category* of X as $\mathsf{WFuk}(X)$.

- **Objects:** The objects will be exact Lagrangian submanifolds, such that outside a compact set they are translation invariant in the cylindrical ends/with respect to the cylindrical structure at infinity ("invariant under Liouville flow" for differential geometers). Morally, this means that we don't want the Lagrangian zig-zagging back and forth as we go to infinity.

- **Morphisms:** We need to control the behaviour of the L_i at infinity so that we can have invariance of the intersections at infinity. For this we use a Hamiltonian perturbation φ_H to move L_0 so that it is in the right position to intersect L_1. Intuitively, φ_H behaves like the identity $\varphi_H \sim \mathrm{id}$ in a compact set, and as a (quadratic; $H \sim r^2$) rotation in the cylindrical ends to infinity. The hom set is

$$\hom(L, \varphi(L)) = \bigoplus_{p \in L \cap \varphi(L)} \mathbb{C} \cdot p.$$

- **Product:** The composition

$$\hom(L_0, L_1) \otimes \hom(L_1, L_2) \to \hom(L_0, L_2)$$

is given by

$$(p, q) \mapsto \sum_r \eta_{pqr} \cdot r \tag{1}$$

where η_{pqr} is the count of holomorphic discs with boundary as in the figure below (in the language of our previous definitions, holomorphic discs correspond to $u \in \mathcal{M}$).

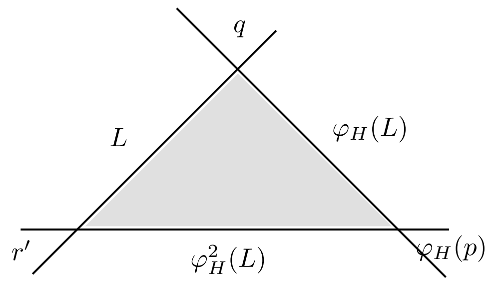

Figure 4: A holomorphic triangle.

The φ_H^2 means that we have perturbed L "double" the amount. The bottom right point is denoted $\varphi_H(p)$ because it is a result of φ_H applied to an intersection between L and $\varphi_H(L)$. The bottom left point is denoted by r', because really, it corresponds 1-1 to a point $r \in L \cap \varphi(L)$ (see Remark 3.9).

Remark 3.9. Note that [1] the intersections $\varphi_H^2(L_0) \cap L_2$ correspond one-to-one to the intersections $\varphi_H(L_0) \cap L_2$.

3.5.1 The case of the cylinder

Denote the Lagrangian $\mathbb{R} \cong \mathbb{R} \times \{\mathrm{pt}\} = \{\theta = 0\}$ in X by L. We would like to work out the endomorphisms of L. As we know, to do this we should perturb L so that it can intersect itself, and by the discussion above, this will require "wrapping" it around the cylinder. Note that

$$\hom(L, L) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot x_n.$$

This has infinite rank which is a good start (since we want to match it with $\mathbb{C}[z^{\pm 1}]$). But to get a true correspondence, we need to compute the multiplication (product) structure $x_n \cdot x_m$, and this requires introducing a second perturbation (therefore third Lagrangian) to our picture (so that we can get a (holomorphic) triangle as in Definition 3.8).

Foreshadowing remark: Keep in mind that we would like for $x_n \in \hom(L, L)$ to correspond to the power $z^n \in \mathbb{C}[z^{\pm 1}]$.

The way we define our product equation (see equation 1) in our simple cylindrical case is as follows. Take an intersection point of $L \cap \varphi_H(L)$ and also an intersection point of $\varphi_H(L) \cap \varphi_H^2(L)$. These points determine a third point (a point of $L \cap \varphi_H^2(L)$) and this resulting point will be our product. Let's see that this works for $x_1 \cdot x_3 = x_4$, with notation as in the diagram below.

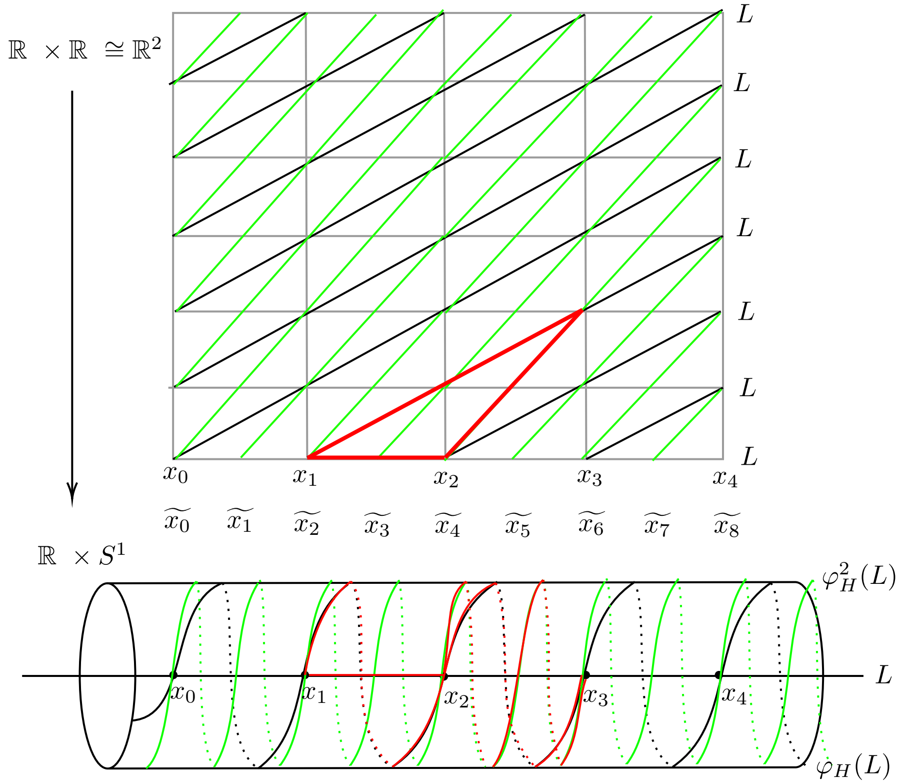

Figure 5: Computing products in the universal cover of X.

Indeed, it is equivalent to work in the universal cover of X for our purposes; in this case the universal cover is the plane $\mathbb{R}^2$. The point x_1 is determined by the intersection of L and $\varphi_H(L)$, and the point x_3 by the intersection of $\varphi_H(L)$ and $\varphi_H^2(L)$. Tracking back the Lagrangians $\varphi_H(L)$ and $\varphi_H^2(L)$ from the point x_3 (the "tip" of the triangle) until we reach x_1 in $L \cap \varphi_H(L)$ forces $\widetilde{x}_4$ to be the remaining point of our triangle. But by Remark 3.9 which gives us the bijective correspondence $\tilde{x}_i \mapsto x_i$, we conclude that $x_1 \cdot x_3 = x_4$. By looking at the universal cover in the figure above, it is easy to extend this reasoning to $x_i \cdot x_j = x_{i+j}$. We therefore see that the product structure of $\mathrm{Hom}^*(L, L)$ is the same as that of $\mathbb{C}[z^{\pm 1}]$ where we have $z^i \cdot z^j = z^{i+j}$, via the identification $x_i \mapsto z^i$. Hence,

$$\mathrm{Hom}^*(L, L) \cong \mathbb{C}[z^{\pm 1}],$$

and we conclude that $L = \mathbb{R}$ is mirror to $\mathcal{O}$.

Recall that right at the end of §2, we computed the composition $\mathrm{Hom}^*(\mathcal{O}, \mathcal{O}) \otimes \mathrm{Hom}^*(\mathcal{O}, \mathcal{O}_a) \to \mathrm{Hom}^*(\mathcal{O}, \mathcal{O}_a)$. We lastly need to do the corresponding computation on the A-side, and see that it matches up correctly. Note the figure below.

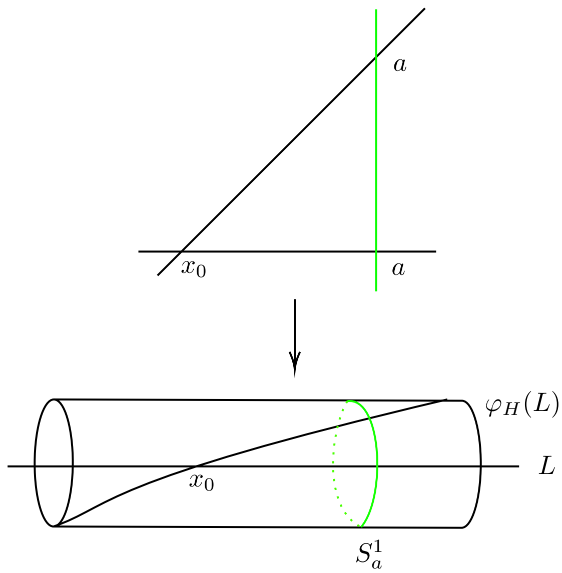

Figure 6: Computing the product associated to $\hom^*(\mathbb{R},\mathbb{R}) \otimes \hom^*(\mathbb{R}, S^1_a) \to \hom^*(\mathbb{R}, S^1_a)$, again in the universal cover $\mathbb{R}^2$ of X.

Consider the composition $\hom^*(\varphi_H(L), L) \otimes \hom^*(L, S^1_a) \to \hom^*(L, S^1_a)$. In other words, this says: take a point in $\varphi_H(L) \cap L$ and a point in $L \cap S^1_a$. This will determine a point in $\varphi_H(L) \cap S^1_a$ (the remaining point of the triangle), except in this situation, notice that the position of the determined "product" point is the same with respect to the x-axis (the Lagrangian L) as the point we took in $L \cap S^1_a$. For our picture, the above procedure corresponds to taking x_0 and a, and then the point that is the product is still a because of the x-axis argument we just gave. In summary, we have (for a general x_i) $x_i \cdot a = a$. Now if we pass from $\hom^*$ to Hom^*, we get something like

$$g(\dots, x_{-1}, x_0, x_1, \dots) \cdot a = g(\dots, a, a, a, \dots) = g(a).$$

This is what we had at the end of §2, so we are done.

Remark 3.10. This HMS example of $X = \mathbb{C}^* = \mathbb{R} \times S^1 = Y$ can be generalised to HMS of $X = Y = (\mathbb{C}^*)^n$ for $n > 1$.

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