The cylinder

B-side

$$Y = \mathrm{Spec}\left(\mathbb{C}[z^{\pm 1}]\right)$$

$\mathcal{O}_a := \mathbb{C}[z^{\pm 1}]/z-a$ (skyscraper sheaf at $z=a$)

$$\mathrm{Hom}^*(\mathcal{O}_a, \mathcal{O}_b) \simeq \begin{cases} 0 & a \neq b \\ \mathbb{C}[\theta]/\theta^2 & a=b, \ |\theta| = 1 \end{cases}$$

$\mathcal{O} := \mathbb{C}[z^{\pm 1}]$ (structure sheaf)

$$\mathrm{Hom}^*(\mathcal{O}, \mathcal{O}) \simeq \mathbb{C}[z^{\pm 1}]$$

$$\mathrm{Hom}^*(\mathcal{O}, \mathcal{O}_a) \simeq \mathbb{C}[0]$$

$$\mathrm{Hom}^*(\mathcal{O}_a, \mathcal{O}) \simeq \mathbb{C}[1]$$

$$\begin{array}{ccccc} \mathrm{Hom}^*(\mathcal{O}, \mathcal{O}) & \otimes & \mathrm{Hom}^*(\mathcal{O}, \mathcal{O}_a) & \longrightarrow & \mathrm{Hom}^*(\mathcal{O}, \mathcal{O}_a) \\ f(z) & \otimes & 1 & \longmapsto & f(a). \end{array}$$

A-side

$X = \mathbb{R} \times S^1, \quad \omega = dr \wedge d\theta = d(r\,d\theta)$

If $\omega = d\alpha$, we say (X, ω) is exact.

If $L \subset X$ Lagrangian, $d(\alpha|_L) = 0$. If $\alpha|_L = dh_L$ we say L is exact.

Defn: The exact Fukaya category of X is like the Fukaya category, except

- Ob = exact Lagrangians
- $hom(L_0, L_1) = \mathbb{C}\langle L_0 \cap L_1 \rangle$
  Not Novikov field
- compositions as before, just set $T = 1$.

Why does this work? Recall we introduced Nov. field to accommodate infinite sums, and it worked by Gromov compactness. Certainly

$$\Lambda \xrightarrow{T \mapsto 1} \mathbb{C}$$

is not well-defined!

Answer: exactness. Given $p \in L_0 \cap L_1$, define

$A(p) := h_{L_0}(p) - h_{L_1}(p)$

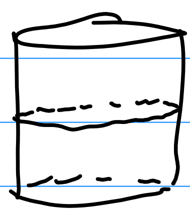
Lem: If $u \in \mathcal{M}(p_0, \ldots, p_k)$, then
$$\omega(u) = A(p_0) - \sum_{i=1}^{k} A(p_i)$$ ← independent of u!

Pf: Stokes.

So only one power of T appears $\Rightarrow$ can set $T=1$.

Since $\mathrm{Fuk}^{ex}(X)$ is $\underline{\underline{\mathbb{C}}}$-linear, we can compare it with $D^b(\mathbb{C}^*)$.

Only compact exact Lagrangian in X, up to Ham. isotopy:



$S^1 = \{r=0\}$

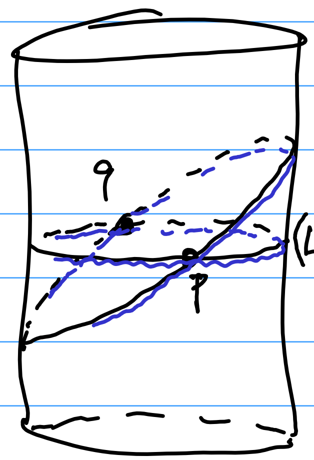
We already saw $\mathrm{Hom}^*(L, L) \simeq \mathbb{C}[\theta]/\theta^2$. S^1 is mirror to $\mathcal{O}_1$.

What about other $\mathcal{O}_a$? Need to introduce a new type of object of $\mathrm{Fuk}(X)$: (L, ξ)
(extra bit of 'brane' structure)

L: Lagrangian, ξ: $\mathbb{C}$-loc. sys

Cf. [Aur 14, Rmk 2.11]

E.g.



a = holonomy of ξ_a

Trivialization of loc. systems over blue areas determines iso

$\hom((L_0, \xi_0), (L_1, \xi_1))$

$\simeq \mathbb{C} \cdot p \oplus \mathbb{C} \cdot q$

$$m_1(p) = q - a^{-1} b \cdot q$$

strip on right

strip on left gets monodromy contribution

$$\Rightarrow \mathrm{Hom}^*((S^1, \xi_a), (S^1, \xi_b)) \simeq \begin{cases} \mathbb{C}[\theta]/\theta^2 & a = b \\ 0 & \text{else} \end{cases}$$

$S^1_a := (S^1, \xi_a)$ is mirror to $\mathcal{O}_a$.

What about $\mathcal{O}$? $\mathrm{Hom}^*(\mathcal{O}, \mathcal{O}) \simeq \mathbb{C}[z^{\pm 1}]$ is infinite-dim'l, how can it be $\mathbb{C}\langle L \cap \varphi(L) \rangle$? L must be non-compact.

Introduce $W\mathrm{Fuk}(X)$ [Aur 14, §4]. Let

$\mathbb{R} := \{\theta = 0\}$.

Compute $\mathrm{Hom}^*(\mathbb{R}, \mathbb{R}) \simeq \mathbb{C}[z^{\pm 1}]$.

$\mathbb{R}$ is mirror to $\mathcal{O}$.

Compute $\mathrm{Hom}^*(\mathbb{R}, \mathbb{R}) \otimes \mathrm{Hom}^*(\mathbb{R}, S'_a) \longrightarrow \mathrm{Hom}^*(\mathbb{R}, S'_a)$

show it matches with B-side.

Generalization: $X = Y = (\mathbb{C}^*)^n$.