

Symplectic (co)homology $A = (x)$ $A = (u) \in$

David Jackson-Hansen

Defn: M compact symplectic manifold with boundary
 $W = \partial M \neq \emptyset$. M is Liouville domain, i.e. $\exists \theta, d\theta = \omega$

$Z =$ Liouville vec. field, $\mathcal{L}_Z \omega = 0$

points strictly outward along boundary.

$\alpha = \theta|_{\partial M}$ contact form on W

$\hat{M} := M \cup W \times \mathbb{R}_+$ completion of M

An isomorphism of Liouville domains

$$\phi^* \theta_1 = \theta_0$$

$$\Rightarrow \text{for } \tau \gg 0, \phi(r, y) = (\tau - f(y), \psi(y))$$

Floer setup:

$$H: M \rightarrow \mathbb{R}$$

Note: by exactness, $A_H = \int_S x^* \theta + \int_S H(x(t)) dt$

$$\partial_s u + \mathcal{J}(\partial_t u - X_H(u)) = 0 \quad (*)$$

$$\lim_{s \rightarrow \pm\infty} u(s, t) = x_{\pm}^{\pm}(t)$$

$$E(u) = A_H(x_-) - A_H(x_+)$$

$$H_s = H_+ \quad s \gg 0$$

$$H_s = H_- \quad s \ll 0$$

$$E(u) = A_{H_-}(x_-) - A_{H_+}(x_+) + \int \partial_s H_s(u) ds \wedge dt$$

$$\partial_s H_s \leq 0$$

$M = \mathbb{R} \times Y$, (Y, α) contact M , $\phi \neq 0$, $M \setminus \{0\} = W$

$$H = h(e^r)$$

$$R = \text{Reeb vec. field} \Rightarrow X_H = h'(e^r) R$$

Key: Reeb orbits of period $T \iff$ period-1 orbits at level $\{T\} \times Y$, where $h'(e^r) = T$.

Issue: non-compactness means solutions can escape to $+\infty$

$$\text{Let } p = e^r u$$

If $\partial_s h_s \leq 0$ then we still have a maximum principle \Rightarrow no bubbles escape to ∞ .

Can also consider isomorphism of Liouville domains $r - f_s(y) \Rightarrow$ exponential decay of

$$(*) \quad \partial = (\omega) h_s \text{ over a region}$$

$$(\ast) \quad A - (Lx) \wedge A = (p) \exists$$

$$\begin{aligned} 0 < \delta & \Rightarrow H = \delta H \\ 0 > \delta & \Rightarrow H \end{aligned}$$

Bad definition of symplectic cohomology:

$$H: \hat{M} \rightarrow \mathbb{R}, H = h(e^{\tau}) \quad \tau \gg 0$$

$$\lim_{\tau \rightarrow \infty} h'(e^{\tau}) = \infty, \quad h''(e^{\tau}) \geq 0$$

$$(M, \omega) \xrightarrow{H} HF^*(H)$$

Problem: degeneracy of orbits (Morse-Bott approach).

$$SH^*(M, H) := HF^*(H)$$

pick up Reeb orbits of $W = \partial M$.

2nd definition:

Pick τ which is not the length of a Reeb orbit.

Let

$$H^{\tau}(e^r, y) = \tau e^r + \beta \quad \text{for } \tau \gg 0$$

picks up orbits of period $\leq \tau$.

Given $\tau_- < \tau_+$, pick a function H_s interpolating between H_{τ_-} and H_{τ_+} .

$$\Rightarrow HF^*(H^+) \rightarrow HF^*(H^-)$$

$$SH^* < \tau_+ \rightarrow SH^* < \tau_-$$

$$\text{Now define } SH^*(M) := \lim_{\tau \rightarrow \infty} SH^* < \tau$$

This agrees with the previous definition.

If we take $\tau < \text{length of shortest Reeb orbit}$, all orbits of H^τ lie in M and are ~~all~~ critical points of H^τ

$$\Rightarrow HF^{* < \tau}(M) \cong H^{*+n}(M)$$

So we have

$$H^{*+n}(M) \xrightarrow{\cong} SH^*(M)$$

Ex: Ball D^{2N} , $\hat{M} \cong \mathbb{C}^N$, 1-form $\rightarrow d^c(\frac{1}{2}|x|^2)$
 $SH^*(D^{2N}) = 0$

Thm (Viterbo): Let $M = D^*N$, the disk bundle w.r.t. some Riemannian metric.

$$SH^*(M) \cong H_*(\mathbb{Z}, \mathbb{Z})$$

Exact Manifolds

Let $L \subset M$ be an exact, compact Lagrangian. Floer's equation for α a half-cylinder with Lag. boundary conditions

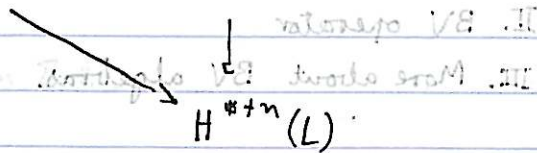
$$u: \mathbb{R}^{\geq 0} \times S^1 \rightarrow \hat{M}, \quad u(\infty, t) = x_+(t) \\ u(0, t) \in L$$

exactness + compactness \rightarrow energy bounds \rightarrow good moduli space.

Use evaluation at $(0,0)$ to get cycle in L for every x closed H -orbit

This gives a map $SH^*(M) \rightarrow H^{*+n}(L)$

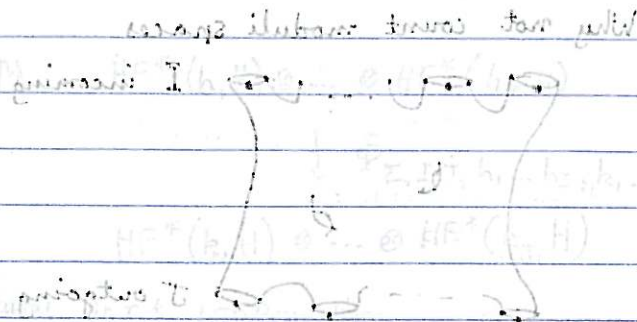
$$H^{*+n}(M) \xrightarrow{\quad} SH^*(M)$$



Thm: $H^{*+n}(M) \rightarrow H^{*+n}(L)$ is the restriction map (when things are regular)

Cor: If M has an exact compact Lagrangian submfld then $SH^*(M) \neq 0$ (since $1 \rightarrow SH^*(M) \rightarrow 1$).

Why true: continuation map with $H_+ = 0$, H_- appropriate, cylinders which can be capped off



$$T^*(H) \xrightarrow{\quad} T^*(H)$$

Claim: ...

$$0 \rightarrow \dots \rightarrow T^*(H) \rightarrow \dots$$