Symplectic (co)homology

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<u>Defn</u>: M compact symplectic manifold with boundary $W = \partial M \neq \phi$. M is Liouville domain, i.e. $\exists \theta$, $d\theta = \omega$

Z = Liouville vec. field, $\iota_Z \omega = \theta$
points strictly outward along boundary.

$\alpha = \theta|_{\partial M}$ contact form on W.

$\hat{M} := M \cup_W W \times \mathbb{R}_+$ completion of M

An isomorphism of ~~such~~ Liouville domains
$\phi : \hat{M}_0 \xrightarrow{\sim} \hat{M}_1$

$\phi^* \theta_1 = \theta_0$

$\Rightarrow$ for $r \gg 0$, $\phi(r, y) = (r - f(y), \psi(y))$

<u>Floer setup</u>:

$H : M \to \mathbb{R}$

Note: by exactness, $A_H = -\int_{S^1} x^* \theta + \int_{S^1} H(x(t))\, dt$

$$\partial_s u + J(\partial_t u - X_H(u)) = 0 \qquad (*)$$

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t).$$

$$E(u) = A_H(x_-) - A_H(x_+)$$

$$H_s = \begin{cases} H_+ & s \gg 0 \\ H_- & s \ll 0 \end{cases}$$

$$E(u) = A_{H_-}(x_-) - A_{H_+}(x_+) + \int \partial_s H_s(u)\, ds \wedge dt$$

$$\partial_s H_s \leq 0$$

$M = \mathbb{R} \times Y$, (Y, α) contact M

$H = h(e^r)$

R = Reeb vec. field $\Rightarrow X_H = h'(e^r) R$

Key: Reeb orbits of period T $\Longleftrightarrow$ period 1 orbits at level $\{r\} \times Y$, where $h'(e^r) = T$.

Issue: non-compactness means solutions can escape to $+\infty$

Let $\rho = e^r u$

$(\Delta \rho \geq 0)$

If $\partial_s h_s \leq 0$ then we still have a maximum principle $\Rightarrow$ no bubbles escape to ∞.

Can also consider isomorphism of Liouville domains $r - f_s(y)$ $\Rightarrow$ exponential decay of h_s over a region.

Bad definition of symplectic cohomology:
$H: \hat{M} \to \mathbb{R}$, $H = h(e^r)$, $r >> 0$.
$\lim_{r\to\infty} h'(e^r) = \infty$, $h''(e^r) \geq 0$

Problem: degeneracy of orbits (Morse-Bott approach).
$SH^*(M, H) := HF^*(H)$
pick up Reeb orbits of $W = \partial M$.

2nd definition:
Pick τ which is not the length of a Reeb orbit.
Let
$$H^\tau(e^r, y) = \tau e^r + \beta \quad \text{for } r >> 0$$
picks up orbits of period $\leq \tau$.

Given $\tau_- < \tau_+$, pick a function H_s interpolating between H^{τ_-} and H^{τ_+}

$$\Rightarrow HF^*(H^+) \longrightarrow HF^*(H^-)$$
$$SH^{*<\tau_+} \longrightarrow SH^{*<\tau_-}$$

Now define $SH^*(M) := \varinjlim_{\tau} SH^{*<\tau}$

This agrees with the previous definition.

If we take $\tau <$ length of shortest Reeb orbit, all orbits of H^τ lie in M and are critical points of H^τ

$$\Rightarrow HF^{*<\tau}(M) \cong H^{*+n}(M)$$

So we have

$$H^{*+n}(M) \longrightarrow SH^*(M)$$

Ex: Ball D^{2N}, $\hat{M} \cong \mathbb{C}^N$, 1-form $-d^c(\frac{1}{2}|x|^2)$.

$SH^*(D^{2N}) = 0.$

Thm (Viterbo): Let $M = D^*N$, the disk bundle w.r.t. some Riemannian metric.

$$SH^*(M) \cong H_{-*}(\mathcal{L}N)$$

Exact Manifolds

Let $L \subset M$ be an exact compact Lagrangian. Floer's equation for a half-cylinder with Lag. boundary conditions

$$u: \mathbb{R}^{\geq 0} \times S^1 \longrightarrow \hat{M}, \quad (*), \quad \lim_{s\to\pm\infty} u(s,t) = x_\pm(t)$$

$$u(0,t) \in L$$

exactness + compactness $\Rightarrow$ energy bounds $\Rightarrow$ good moduli space.

Use evaluation at $(0,0)$ to get cycle in L for every x closed H-orbit

This gives a map $SH^*(M) \to H^{*+n}(L)$.

$$\begin{tikzcd} H^{*+n}(M) \arrow[r] \arrow[dr] & SH^*(M) \arrow[d] \\ & H^{*+n}(L) \end{tikzcd}$$

<u>Thm</u>: $H^{*+n}(M) \to H^{*+n}(L)$ is the restriction map (when things are regular)

<u>Cor</u>: If M has an exact compact Lagrangian submfd then $SH^*(M) \neq 0$ (since $1 \to SH^*(M) \to 1$).

Why true: continuation map with $H_+ = 0$, H_- appropriate, cylinders which can be capped off