Orientations in LCH

Georgios

$L \hookrightarrow \mathbb{C}^n \times \mathbb{R}$, $(dt - p_i dq^i)$ Legendrian.

$\mathcal{A} = \mathbb{Z}[H_1(\alpha)]\langle Q \rangle$ Q = Reeb chords on L.

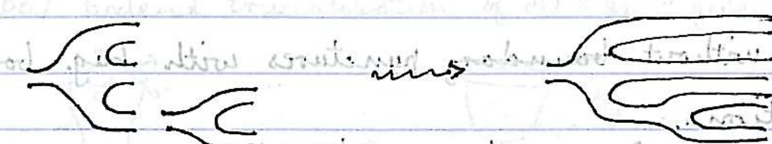
Goal: orient the following moduli spaces,

$\partial^2 = 0$ orient dim 0, 1 moduli spaces

invariance dim 0, −1 1-param families

in a coherent way w.r.t. gluing.

E.g.



$\ker \bar\partial_1$ $\ker \bar\partial_2$ $\mathbb{R} \oplus \ker \bar\partial_1 \oplus \ker \bar\partial_2$

Start with case of closed curves:

$u : \Sigma \xrightarrow[J\text{-hol}]{} M$

Linearisation of $\bar\partial$: $D_u : T_u\mathcal{B} \longrightarrow \Omega^{0,1}(u^*TM)$

$\Gamma(u^*TM)$

$(D_u \xi)(v) = \nabla_v \xi + J\nabla_{jv}\xi + \underbrace{A(\xi)}_{\in \Omega^{0,1}}(v)$

Cauchy-Riemann type operator

CR operators form a convex space$_{\wedge}$ (CR). We want to orient

$T_u\mathcal{M} = \{\xi \in \ker D_u\}$

Consider the determinant line $\det(\mathcal{CR}) \to \mathcal{CR}$ with fibre over F

$$\left(\Lambda^{max} \ker F\right) \otimes \left(\Lambda^{max} \operatorname{coker} F\right)^*.$$

Fact: $\det(\mathcal{CR})$ is orientable.

We may homotope a given $F \in \mathcal{CR}$ to an op. which is $\mathbb{C}$-linear in ξ.

$\Rightarrow$ may use complex orientation for ker, coker.

$\Rightarrow$ get this trivialisation of $\det(\mathcal{CR})$.

Disks without ~~boundary~~ punctures with Lag. boundary conditions.

Lin: $\bar{\partial}_\lambda : \mathcal{H}^2_\lambda(D^2, \mathbb{C}^n) \to \mathcal{H}^1(D^2, \mathbb{C}^n \otimes T^{*0,1}(D^2))$

$\lambda : \partial D \longrightarrow \mathrm{Lag}_0(\mathbb{C}^n)$

oriented Lagrangians in $\mathbb{C}^n$.

$\mathcal{H}^2_\lambda(D^2, \mathbb{C}^n)$ = sections satisfying $\xi(t) \in \lambda(t)\ \forall t \in \partial D$.

(Fooo) There are families of the above boundary conditions over which the det line is non-orientable.

We must use a trivialised boundary condition

$$\lambda(t) : \partial D \to \mathcal{U}(n), \quad \xi(t) \in \lambda(t)\mathbb{R}^n \quad t \in \partial D$$

Lem: The det bundle is canonically orientable over $\Omega(\mathcal{U}(n))$.

<u>Pf</u>: $\pi_1(\Omega(U(n)), A) \cong \mathbb{Z}$ can be generated by $e^{is}A$

Take a basis $v_1, \dots, v_r$ at $\ker \bar\partial_A$

$w_1, \dots, w_s$ at $\operatorname{coker} \bar\partial_A$

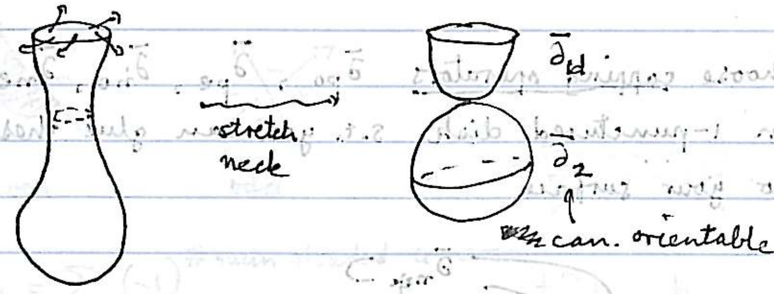
Monodromy of det. line around generator of π_1:

$$\left(e^{is} v_1 \wedge \dots \wedge e^{is} v_r\right) \otimes \left(e^{is} w_1 \wedge \dots \wedge e^{is} w_s\right)^* \Big|_{s=2\pi}$$

$$= (v_1 \wedge \dots \wedge v_r) \otimes (w_1 \wedge \dots \wedge w_s)^*$$

$\Rightarrow$ orientable.

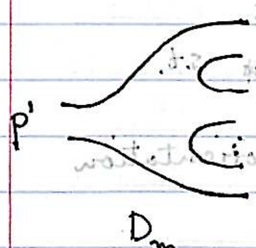
(FOOO) Extend trivialisation of $\partial D \times \mathbb{C}^n$ given by A, inwards.



$\ker \bar\partial_{id} = \mathbb{R}^n$ $\underset{\text{gluing}}{\Longrightarrow}$ canonical orientation on $\bar\partial_A$. $\square$

$\operatorname{coker} \bar\partial_{id} = 0$

<u>Punctured disks w/ Lagr. bdry cond.</u>



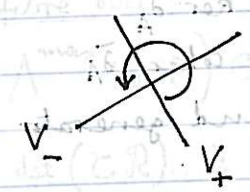
$\lambda : \partial D_m \to U(n)$

given by TL away from punctures

let $V^i_\pm = \lim_{t \to p^i_\pm} \lambda(t)\mathbb{R}^n$.

consider transversal case $V^i_+ \cap V^i_- = 0$.

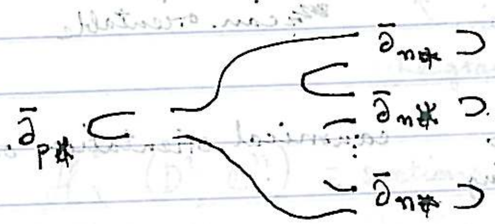
p pos. puncture : p even



if the 'long' rotation $V_+ \rightsquigarrow V_-$ preserves orientation.

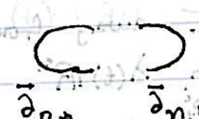
p neg. puncture: p even if (same with $V_+ \leftrightarrow V_-$).
otherwise, odd.

1) Choose <u>capping operators</u> $\bar{\partial}_{po}$, $\bar{\partial}_{pe}$, $\bar{\partial}_{no}$, $\bar{\partial}_{ne}$ on 1-punctured disk s.t. you can glue these to your surface



$\Rightarrow$ trivialised Lagr. bdry. cond.

2) Choose orient. for $\overset{\text{det}}{\bar{\partial}_{n*}}$, orient $\overset{\text{det}}{\bar{\partial}_{p*}}$ s.t.



$\overset{}{\underset{\text{glue}}{\rightsquigarrow}}$ canonical orientation

3) Orient det$(\bar{\partial}_\lambda)$ s.t. gluing capping op induces can. orientation.

Need to choose a spin structure for L: $\Leftrightarrow$ trivialise $TL \oplus \mathbb{R}^2$ over the 1-skeleton s.t. it extends over the 2-skeleton.

· homotope bdy of hol. disks to 1-skel rel. punctures :

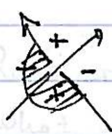
$\Rightarrow$

$$\partial a = \sum_{\dim \mathcal{M}=0} (-1)^{(n-1)(|a|+1)} \left| \mathcal{M}_A(a; \bar{b}) \right| A \bar{b}.$$

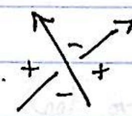
3-dim case

Lie gp spin structure on S^1.

Choosing capping operators against orientation.



even



odd

$$\partial a = \sum (-1)^{\#\text{ non shaded corners}} t^m b_1 \dots b_n$$

$$\mathbb{Z}[H_1(S^1)] = \mathbb{Z} t^n.$$