Neighbourhood Theorems

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Moser's Trick

Q: Suppose there is a homotopy of $\omega_t \in \Omega^p(W)$ $t \in [0,1]$. Is there an isotopy $\phi_t : W \to W$ s.t. $\phi_t^*(\omega_0) = \omega_t$ $\forall t$?

A: Not always. It suffices to find a t-dependent vec. field v_t s.t. $\phi_t = e^{t v_t}$.
i.e. $\dot{\phi}_t(x) = v_t(\phi_t(x))$, $\forall x \in W$.

Differentiate:

$$\phi_t^* \omega_0 = \omega_t$$
$$\Rightarrow \mathcal{L}_{v_t} \omega_t = \dot{\omega}_t$$

Prop: Solution to $\mathcal{L}_{v_t} \omega_t = \dot{\omega}_t$ for exact homotopy of sympl. forms.

Let $\omega_t = \omega_0 + d\alpha_t$

be a homotopy of sympl. forms on W and

$$v_t = \iota_{\omega_t}^{-1}(\dot{\alpha}_t) \qquad (\text{i.e., } \iota_{v_t}\omega_t = \dot{\alpha}_t)$$

Then $\mathcal{L}_{v_t} \omega_t = \dot{\omega}_t$

Pf: $\mathcal{L}_{v_t} \omega_t = d(\iota_{v_t} \omega_t) = d(\dot{\alpha}_t) = \dot{\omega}_t$.

Symplectic neighbourhood theorems

(W, ω) symp. mfd.

Prop (Thurston): If $v: X \to W$ is a sympl. vec. bundle, then $\exists$ sympl. structure $\tilde{\omega}$ s.t. $\tilde{\omega}|_W = \omega$ and $\tilde{\omega}|_{fibre}$: & coincides with the linear sympl. struc.
near the 0-section

Pf: By def'n, $\exists$ closed 2-form η s.t. $\eta|_{fibre}$ defines the linear sympl. structure.
Let $\tilde{\omega} = \eta + v^*\omega$. This is sympl. in a nbhd of 0-section.

Weinstein's thm: Any isotropic immersion $(L \to (W, \omega))$ extends to an isosymplectic immersion.
where U is a nbhd of the 0-section of (T^*L, ω_0)

Pf: Suppose $\dim L = k$. $\exists$ transverse isotropic plane field Θ s.t. $L \oplus \Theta$ is symplectic. Moreover, the space of such k-planes is contractible
($J(TL)$ for J an a.c. struc.)

$\Rightarrow$ the immersion $L \to W$ extends in a homotopically unique way to an immersion $h: U \to W$.
Then $h^*\omega = \omega_0$ on the 0-section of $T^*L \to L$ and $h^*\omega = \omega_0 + d\alpha$ where α vanishes on L.
Apply Moser's trick.

Symplectic nbhd thm:
Let $f: (V, \omega_V) \to (W, \omega_0)$ be an isosympl. immersion, and

$E \to V$ sympl. vector bundle s.t. the fibre over every point $x \in V$ is $df_p(TV)^{\perp \omega_0}$.
Then f extends to an isosymplectic immersion

$(\mathrm{Op}\,V, \omega_E) \longrightarrow (W, \omega_0)$

open nbhd of 0-section.

Pf: By definition, $\exists$ an immersion $\hat{f} : \mathrm{Op}\,V \to W$ extending f s.t.

$$\hat{f}^* \omega_W = \omega_E|_V$$

on the 0-section of $E \to V$ and

$\hat{f}^* \omega_W - \omega_E = d\alpha$ where α vanishes on $TE|_V$.

Apply Moser's trick.

Contact Neighbourhood Thms

$(M, \xi = \ker \alpha)$ contact mfd.

$d\alpha$ is a symplectic structure on ξ.

$L \subset (M, \xi)$ isotropic submfd, i.e. $TL \subseteq (TL)^{\perp d\alpha}$

Rmk: The symplectic structure $d\alpha|_\xi$ does not depend on the choice of α up to conformality i.e. if $f\alpha$ is another contact form then

$$d(f\alpha)|_\xi = f\, d\alpha|_\xi .$$

Defn: A conformal symp. normal bundle $CSN(L)$

$:= (TL)^{\perp} / TL$

Splitting of the ~~tangent~~ normal bundle

$$NL \cong \underbrace{TM/_{\xi}}_{\text{trivialised by Reeb vec. field.}} \oplus \xi/_{(TL)^{\perp}} \oplus CSN(L)$$

Thm: Let $(M_i, \xi_i)_{i=0,1}$ be two contact manifolds with closed isotropic submfds L_i. Suppose there exists an isom. $\Phi: CSN(L_0) \to CSN(L_1)$ covering a diffeo $\psi: L_0 \to L_1$. Then ϕ extends to a contactomorphism

$$\begin{array}{ccc} \psi: Op(L_0) & \to & Op(L_1) \\ \cap & & \cap \\ N(L_0) & & N(L_1) \end{array}$$

s.t. $d\psi|_{CSN(L_0)}$ and Φ are bundle homotopies.

Cor: Diffeomorphic Legendrian submfds admit contactomorphic nbhds.

Ex: $(M^3, \xi) \supset S^1$ Legendrian

$\alpha = \cos\theta\, dx - \sin\theta\, dy$