Morse - Bott Theory

(X,g) - closed Riemannian n-manifold

$f: X \to \mathbb{R}$ is Morse iff all its critical points are non-degenerate

$$\left(\Rightarrow f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2 \right. \quad \text{in appropriate coordinates near } p \in \operatorname{crit} f$$

$$k := \text{Morse index of } p := \mu_M(p) \Big)$$

E.g. $X = S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$ $\quad f(x) = x_1$

Morse complex $C_M^*(X, f, g) = (\mathbb{R}\langle \operatorname{crit} f \rangle, d_M)$

where $\langle q, d_M p \rangle :=$ coefficient of q in $d_M p := \# \mathcal{M}_M^f(p,q)$

where $\mathcal{M}_M^f(p,q) = \left(\int_p^q \nabla f \right)$ flowlines of ∇f and $\# :=$ ~~num~~ signed count of 0-dimensional components.

when this ~~spa~~ moduli space is regular (Morse - Smale).

Thm: $C_M^*(X) \underset{q.i.}{\simeq} C^*_{\text{singular}}(X)$. $\quad \Rightarrow H_M^*(X) \cong H^*_{\text{sing}}(X)$.

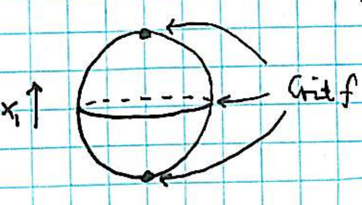
Defn: $f: X \to \mathbb{R}$ is Morse - Bott if $\operatorname{Crit} f$ is a disjoint union of smooth submanifolds of X, and f is nondegenerate in directions normal to $\operatorname{Crit} f$.

$$\left(\Rightarrow f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{\ell} x_i^2 \right.$$

$$\operatorname{Crit} f = \{x_{\ell+1} = \dots = x_n = 0\} \quad \text{in appropriate coordinates near } p \in \operatorname{crit} f \Big)$$

E.g. ~~$S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$~~ $f(\underline{x}) = x_1^2$

$X = S^2$ as before



Often functions with some symmetry are Morse - Bott.

Can still use f to compute $H^*(X)$: for each component C_j of crit f, choose a tubular neighbourhood N_j of C_j and a C^∞ function $\tilde{h}_j : X \to \mathbb{R}$ s.t.

- $\operatorname{supp} \tilde{h}_j \subset N_j$
- $h_j := \tilde{h}_j|_{C_j}$ is Morse
- $\tilde{h}_j$ is constant in normal directions to C_j,

Define $f_\varepsilon : X \to \mathbb{R}$, for $\varepsilon > 0$, by

$$f_\varepsilon := f + \varepsilon \sum_j \tilde{h}_j .$$

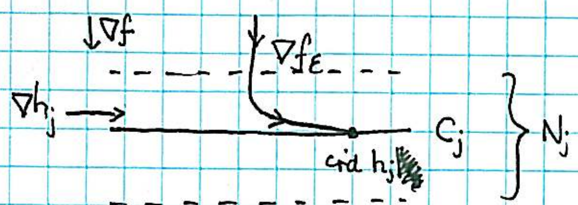
$$\left(\Rightarrow f_\varepsilon(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{\ell} x_i^2 + \varepsilon h_j(x_{\ell+1}, \dots, x_n) \quad \text{near } p \in C_j\right).$$

~~Now study~~ Now for suff. small $\varepsilon > 0$, f_ε is Morse with $\text{crit } f_\varepsilon = \bigcup_j \{\text{crit } h_j\}$. Assume Morse-Smale, then

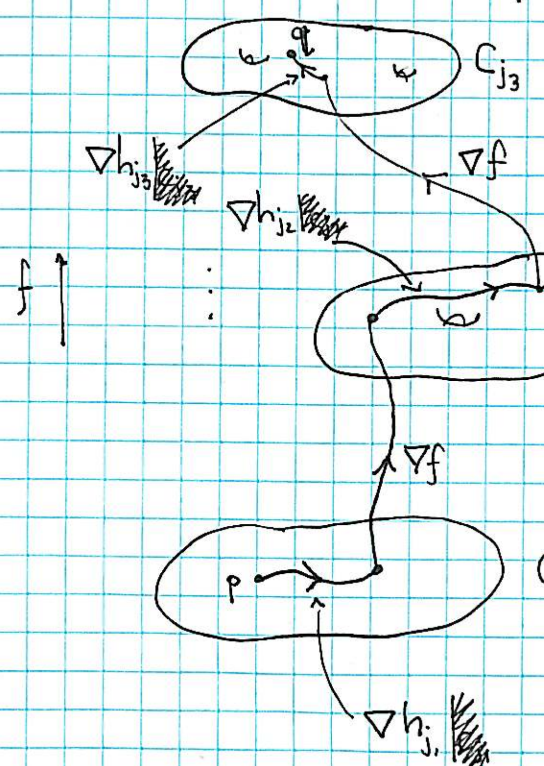
$H^*_{sing}(X) \cong H^*_M(X, f_\varepsilon, g)$. take the limit $\varepsilon \to 0$.

Also note $\mu_M(p) = k + \mu_{M, h_j}(p)$.

Away from C_j, $\nabla f_\varepsilon = \nabla f$. Near C_j, $\nabla f_\varepsilon = \nabla f + \varepsilon \nabla \tilde{h}_j$



<u>Defn:</u> A Morse-Bott trajectory from p to q looks like this:



Call the moduli space of these objects $\mathcal{M}_{MB}(p, q)$. <u>Prop:</u> (Banyaga) For small enough $\varepsilon > 0$, it is isomorphic to $\mathcal{M}_M^{f_\varepsilon}(p, q)$.

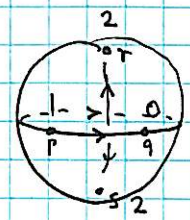
<u>Define</u> the Morse-Bott complex $C^*_{MB}(X, f, g, h_j) = (\mathbb{R}\langle \bigcup_j \text{crit } h_j \rangle, d_{MB})$

where $\langle q, d_{MB}\, p \rangle := \# \mathcal{M}_{MB}(p, q)$.

<u>Thm 2:</u> $H^*_{MB}(X) \cong H^*_{sing}(X)$.

($H^*_{MB}(X) \cong H^*_M(X, f_\varepsilon)$ for small enough $\varepsilon > 0$, so follows from Thm 1)
by Prop (coefficients the same)

E.g. $X = S^2$, $f(x) = x_1^2$:



$$d_{MB}(p) = q - q = 0$$

$$d_{MB}(q) = r + s$$

$$d_{MB}(r) = d_{MB}(s) = 0$$

$$\Rightarrow H^*_{MB}(S^2) \cong \langle p, \underset{-s}{\overset{}{r}} \rangle$$

<u>Symplectic Homology</u>: assume we define using $H: \hat{M} \to \mathbb{R}$ with $\frac{\partial H}{\partial r} \to \infty$.

We're studying Morse theory of the action functional on the loop space:

$$\mathcal{A}_H(\gamma, [\sigma]) := -\int_\sigma \omega - \int_{S^1} H(t, \gamma(t))\, dt$$

homotopy class of disks bounding γ

where $\gamma: \mathbb{R}/\mathbb{Z} \to \hat{M}$ $\quad H: \mathbb{R} \times \hat{M} \to \mathbb{R}$.

Suppose $H: \hat{M} \to \mathbb{R}$, autonomous Hamiltonian, (i.e. choose a Morse function on $\hat{M}$).

Then $\quad \mathrm{Crit}\, \mathcal{A}_H = \{\gamma: S^1 \to \hat{M},\ \dot{\gamma} = X_H\}$
Ham

"Morse flow" of $\mathcal{A}_H$ is given by $\quad u: \mathbb{R} \times S^1 \to \hat{M}$

$$\partial_s u + J_t(\partial_t u - X_H) = 0.$$

Note $\mathcal{A}_H$ is invariant w.r.t. the S^1 action rotating the loop γ

$\Rightarrow$ critical points of $\mathcal{A}_H$, which are not constant loops, appear in S^1 families.

The assumption that orbits of X_H are non-degenerate (transverse return map $\Rightarrow$ no nearby orbits of X_H) ensures that $\mathcal{A}_H$ is "Morse-Bott", i.e. non-degenerate in transverse directions to S^1 critical manifold.

So, apply Morse-Bott techniques.

- Choose a (non-degenerate) Morse function $H: \widehat{M} \longrightarrow \mathbb{R}$.
- For each ~~[illegible]~~ orbit γ_j of X_H, ~~choose~~ let S_{γ_j} be its image (~~it may be~~ a multiple cover). Choose a Morse function $h_j : S_{\gamma_j} \to \mathbb{R}$. ~~Choose a Morse function $h_j : S_{\gamma_j} \to \mathbb{R}$.~~
  suppose it is not

Now ~~[illegible]~~ symplectically trivialise a neighbourhood N_j of S_{γ_j}, so it is ~~[illegible]~~ symplectomorphic to a neighbourhood of $(\theta, 0, \dots, 0)$ in $\left(T^*\mathbb{R}^n /_{(\theta, \underline{v}) \sim (\theta+1, \underline{v})}, \omega_{std} \right)$.

~~Extend [illegible]~~ In these coordinates, define $\tilde{h}_j : \widehat{M} \times \mathbb{R} \to \mathbb{R}$ so that

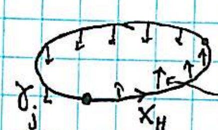
- $\operatorname{supp} \tilde{h}_j \subset N_j$
- ~~[illegible]~~ $\tilde{h}_j(\theta, \underline{v}, t) = h_j(\theta - t)$ near S_{γ_j}
- ~~$\tilde{h}_j = h_j(\theta)$ near S_{γ_j}~~ (if γ_j is an ℓ_j-fold cover of S_{γ_j}, then $h_j(\theta - \ell_j t)$).

Now let

$$H_\varepsilon(\theta, x) := H(x) + \varepsilon \sum_j \tilde{h}_j$$

Prop: $\operatorname{Crit}(\mathcal{A}_{H_\varepsilon}) = \operatorname{crit} H \cup \bigcup_j \operatorname{crit} h_j$.

Picture:

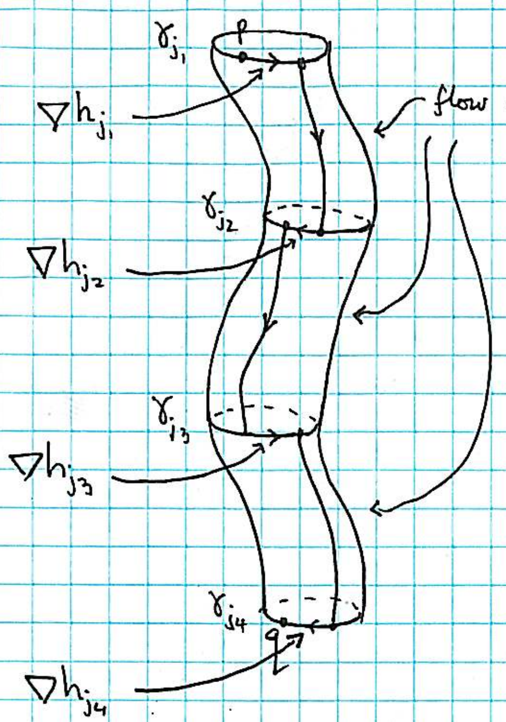


perturbation $= \varepsilon X_{\tilde{h}_j}$ $\left(= \varepsilon J (\nabla \tilde{h}_j) \right)$

Perturbation rotates with γ_j $\Rightarrow$ can't return to starting point unless it starts on $\operatorname{crit}(h_j)$.

~~Defn: A Morse-Bo~~

Defn: A Morse-Bott Floer trajectory from p to q looks like this:



flow of $\mathcal{A}_H$ = solution of $\partial_s u + J_t(\partial_t u - X_H) = 0$

Call the moduli space of these $\mathcal{M}(p, q, J, H, \{h_\gamma\})$.

Prop: For $\varepsilon > 0$ small enough, $\mathcal{M}(p, q, J, H, \{h_\gamma\}) \cong$ moduli space of Floer trajectories w.r.t. H_ε from p to q.

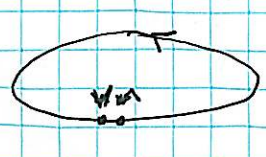
Cor: We can compute $SH^*(M)$ using ~~Morse~~ counts of Morse-Bott Floer trajectories.

~~Observation~~: If we choose h_j to have ~~one~~ two critical points, ~~one close "arbitrarily close to each other"~~, we can ~~get away with defining~~ define

$BSH^*(M)$

to be generated by Crit H together with generators $\hat{\gamma}$ (max), $\check{\gamma}$ (min) for each orbit γ of X_H, having indices
$$\mu(\hat{\gamma}) = \mu_{CZ}(\gamma)$$
$$\mu(\check{\gamma}) = \mu_{CZ}(\gamma) + 1.$$

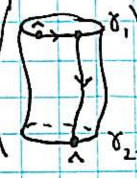
Choose



~~Then d_{BH^*} counts~~ Then $d_{BH^*}(\hat{\gamma}_1)$ counts:

- $\langle p, d_{BH}\hat{\gamma}_1\rangle = 0$ for generic H $\Bigg($ 

has dim ≥ 1 $\Bigg)$

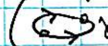
- $\langle \hat{\gamma}_2, d_{BH^*}\hat{\gamma}_1\rangle = \#\Bigg($ 

$\Bigg) = \#\Bigg($ 

$\Bigg)$

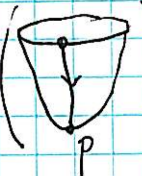
- $\langle \check{\gamma}_2, d_{BH}\hat{\gamma}_1\rangle = 0$ if $\gamma_1 \neq \gamma_2$ $\Bigg($ $\#$ 

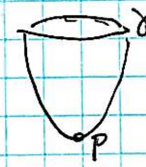
has dim ≥ 1 $\Bigg)$

  $\langle \check{\gamma}_1, d_{BH}\hat{\gamma}_1\rangle = \begin{matrix} 2 & \text{bad} \\ 0 & \text{good} \end{matrix}$ $\Big($ 

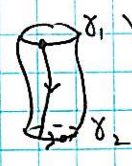
with appropriate signs $\Big)$

$d_{BH}(\check{\gamma}_1)$ counts:

- $\langle p, d_{BH}\check{\gamma}_1\rangle = \#\Bigg($ 

$\Bigg) = \#\Bigg($ 

$\Bigg)$

- $\langle \check{\gamma}_2, d_{BH}\check{\gamma}_1\rangle = \#\Bigg($ 

$\Bigg) = \#\Bigg($ 

$\Bigg)$

- $\langle \hat{\gamma}_2, d_{BH}\check{\gamma}_1\rangle = \#\Bigg($ 

$\Bigg)_{0\text{-dim}} + \#\Bigg($ 
$\Bigg)_{0\text{-dim}}$

cylinders that match marked points on orbits

pairs of cylinders s.t. marked points from top, bottom, and central orbit are in correct order.

$\langle q, d_{BH}(p)\rangle = \#$ 

can arrange all ~~~~ $\check{\gamma}$, $\hat{\gamma}$ have higher energy $\Rightarrow$ don't contribute to $d_{BH}(p)$.