Legendrian contact homology

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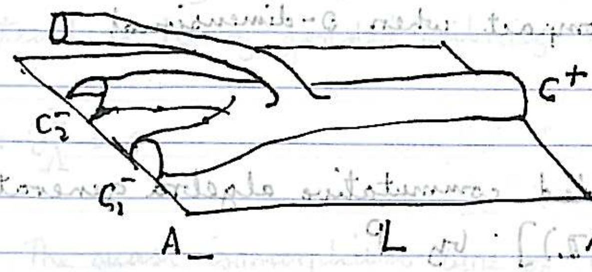
We're counting curves with boundary looking like

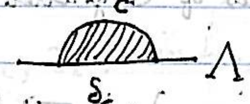


We assume no c Reeb chord is part of a Reeb orbit (generic).

Assume $w_2(L) = 0$ (to orient moduli spaces).

Cap off all $\gamma \in P$ with surface F_γ, $\partial F_\gamma = \gamma$.

Similarly cap c with discs:



Moduli spaces:

$D \subseteq \mathbb{C}$ unit disk

$Z = \{z^+, z_1^-, \dots, z_\sigma^-\}$

boundary punctures ordered w.r.t. bdry orientation

$X = \{x_1^-, \dots, x_s^-\}$

ordered set of interior boundary punctures with asymptotic markers.

$A \in H_2(W)$

$C = \{c_1^-, \dots, c_\sigma^-\} \subseteq \mathcal{C}$ (set of Reeb chords) neg. chords

$c^+ \in \mathcal{C}$ pos. chord

$\Gamma = \{\gamma_1^-, \dots, \gamma_s^-\} \subseteq \mathcal{T}$

neg. orbits

$\mathcal{M}^A(c, c^+, J, W, \Lambda, J) := \{J\text{-hol. curves } D \setminus (Z \cup X), \partial(D \setminus (Z \cup X)) \longrightarrow \Lambda \times \mathbb{R}\}$

Rmk: $\mathcal{M}/\mathbb{R}$ compact when 0-dimensional

Homology

$\mathcal{V} = \mathbb{Z}_2$-graded commutative algebra generated over $\mathbb{Q}[H_2(V, \mathbb{Z})]$ by $\mathcal{P}$

$\partial =$ as in last talk.

$K := K(V, \Lambda, \alpha) =$ graded associative algebra over $\mathcal{V}$ generated by $\mathcal{C}$

Elements of $\mathcal{C}$ graded by Maslov #'s.

Define $\partial_\Lambda : K \longrightarrow K$ by

$$\partial_\Lambda(c) := \sum \frac{\#}{k!\, \prod k_j^{\gamma_j}\, i_j!} c_1 \ldots c_\sigma \gamma_1^{i_1} \ldots \gamma_k^{i_k} e^A$$

sum over $A \in H_2(V, \mathbb{Z})$

$(c_1, \ldots, c_\sigma)$ $(\gamma_1 \ldots \gamma_k)$

$(i_1, \ldots, i_k)$

$k_j =$ mult. of γ_j

$$\# := \begin{cases} 0 & \text{if } \dim \mathcal{M}/\mathbb{R} \neq 0 \\ \text{alg \# of components of } \mathcal{M}/\mathbb{R} & \text{else} \end{cases}$$

$$\partial_\Lambda(\gamma) := \partial(\gamma)$$

boundary on $\mathcal{U}$, $\gamma \in \mathcal{U}$

extend to K by graded Leibnitz rule.

<u>Prop</u>: $\partial_\Lambda^2 = 0$

<u>Prop</u>: The quasi-isomorphism type of (K, ∂_Λ) depends only on ξ and Legendrian isotopy type of Λ. i.e., if you have Λ_t, α_t, J_t; $t \in [0,1]$ an (admissible) homotopy, there is an induced homomorphism

$$\Phi : (K(V, \Lambda_0, \alpha_0), \partial_{\Lambda_0}) \rightarrow (K(V, \Lambda_1, \alpha_1), \partial_{\Lambda_1})$$

- homotopic data to $(\Lambda_t, \alpha_t, J_t)$ induces a homotopy of corresponding ~~homo~~ $\Phi \Rightarrow$ induce same map in homology.
- Composition of connecting homotopies induces composition of Φ.

<u>Prop</u>: $H(K, \partial_\Lambda) = \ker \partial_\Lambda / \operatorname{im} \partial_\Lambda$ has the structure of a module over contact homology $HC(V, \xi)$ and this structure is an invariant of Λ.

Example

Suppose $P = \phi$

The space of trajectories of R_α is a manifold

$M = V/\mathbb{R}$

E.g. $V = (T^* \mathbb{R}^n) \times \mathbb{R}$

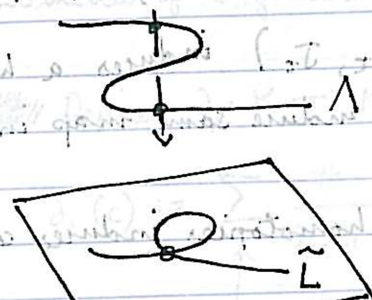
$M = T^* \mathbb{R}^n$

1. M inherits a symplectic structure ξ

2. The projection $\pi: V \to V/\mathbb{R}$ takes $\Lambda \subseteq V$ to an immersed Lagrangian

$\tilde{L} \subseteq M$

s.t. the self-intersections of $\tilde{L}$ are transverse.



3. These intersection points correspond to Reeb chords in V.

In particular, K is generated by the self-intersections of $\tilde{L}$.

4. We can find compatible a.c. structures on $V \times \mathbb{R}$, $V/\mathbb{R}$

s.t. $\begin{tikzcd} V \times \mathbb{R} \arrow[r] \arrow[rr, bend right, "\text{holomorphic}"'] & V \arrow[r] & V/\mathbb{R} \end{tikzcd}$

So J-holom. curves in W descend to J_M-holo curves in M.

<u>Exercises:</u> (Tobias)

1. Lagrangian Floer homology

L_0, L_1

$CF = \mathbb{Z}_2 \langle L_0 \pitchfork L_1 \rangle$

$$\partial x = \sum_{\dim \mathcal{M}(x,y)=1} \#(\#\mathcal{M})\ y$$

Also use: $M_0 = M \subset T^*M$

$M_1 = \Gamma(df) \subset T^*M$

$HF_*(M_0, M_1) \cong H_*(M)$.

exact Lag. $\subset \mathbb{C}^n$

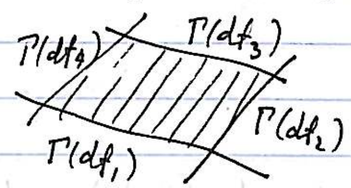
Prove Gromov's thm: no exact Lag's in $\mathbb{C}^n$.

Use: $HF_*(L_0, L_1) \cong HF_*(\Phi_H(L_0), L_1)$ (Hamiltonian isotopy invariance)

2. T^*M, $f_1, \dots, f_k$ Morse functions on M

$\Gamma(df_j) \underset{Lag}{\subset} T^*M$

$\Gamma(df_j) \cap \Gamma(df_\ell) = \mathrm{crit}(f_j - f_\ell)$



what is the dimension of moduli space of these polygons?

$T^*M \times \mathbb{R} = J^1(M)$

3. Compute $CH(S^{2n-1}, \xi_{st})$

(Reeb field$_x = i \cdot x$)

4. (after Albin's talk) compare Legendrian homology of knot in $\mathbb{R}^{2n-1}$ with that of the same knot in a Darboux chart in S^{2n-1}.