Intro to Symplectic & Contact Geometry

Vinicius Gripp

Symplectic Manifold

M^{2n}, ω = non-degenerate closed 2-form

$\Updownarrow$ ω^n = volume form $\Updownarrow$ $d\omega = 0$

E.g. $\mathbb{R}^{2n}$, $\omega_0 = \sum_i dp_i \wedge dq^i$

Darboux: $(M, \omega) \underset{\text{locally}}{\cong} (\mathbb{R}^{2n}, \omega_0)$

Contact Manifold

Y^{2n+1}, $\xi \subset TY$ hyperplane distribution.

"maximally non-integrable"

$\Updownarrow$

locally $\xi = \ker \alpha$; $\alpha \wedge (d\alpha)^{n-1}$

On a closed $2n$-manifold, the symplectic form can't be exact (ω^n is an exact volume form).

If M is open, $\omega = d\alpha$ can be exact.

In this situation, we define the Liouville vector field X by

$$\omega(X, \cdot) = \alpha$$

$$\Rightarrow \mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$$
$$= d\alpha$$

Weinstein Manifold

Exhausting Morse function $f: M \to \mathbb{R}$ (proper, bounded below). X (the Liouville vector field) is gradient-like for f, i.e. $df_p(X) > 0$ at p not a critical point.

<u>Cobordisms</u>

(M, ω) - symplectic

$\partial M = Y_1 \cup (-Y_0)$ - contact manifolds

Exact cobordism: $\omega = d\alpha$,

$\alpha|_{Y_1, Y_0}$ is a contact form

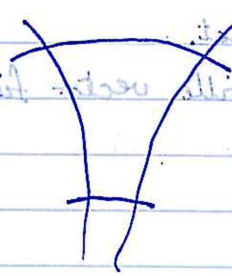
X points outward on Y_1

inward on Y_0.

<u>Symplectisation</u>

Y^{2n-1} $\quad \alpha$ contact form

$(Y \times \mathbb{R}_t, d(e^t \alpha))$ is exact symplectic: the <u>symplectisation</u> of (Y, α).



If we restrict to an interval in $\mathbb{R}$, we get a cobordism.

Given an exact cobordism, we can glue parts of the symplectization onto the ends to make a complete exact symplectic manifold.

Defn: almost-complex structure.

(M^{2n}, ω)

$J: TM \to TM \quad J^2 = -\mathbb{1}$

A J-holomorphic curve is a map $u: (\Sigma, j) \to (M, J)$

$du \circ j = J \circ du$

Defn: Given (Y^{2n-1}, α) contact, the Reeb vec. field, R, satisfies

$d\alpha(R, \cdot) = 0$

$\alpha(R) = 1$

If γ is an orbit of the Reeb ~~vec field~~ vector field, and J is an a.c. structure on the symplectization which preserves ξ and satisfies

$J\left(\frac{\partial}{\partial t}\right) = R$

and is compatible with ω (i.e. $\omega(J\cdot, \cdot)$ is a metric).