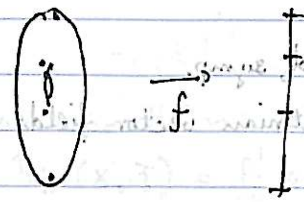
Floer Homology

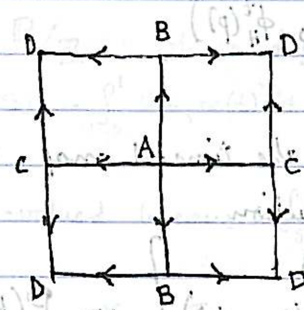
Ramon

Morse Homology



nothing interesting happens between critical points of f.

E.g. On T^2:



$\mathcal{M}(A, B)$:= moduli space of flowlines of ∇f from A to B$/\mathbb{R}$
= {2 pts}

$\mathcal{M}(A, C) = \{2 \text{ pts}\}$

$\mathcal{M}(A, D) = \{4 \text{ intervals}\}$

$\dim \mathcal{M}(p, q) = \text{ind } p - \text{ind } q - 1$

Ingredients of a Floer Homology:

- transversality
- orientation
- compactification
- gluing

Gluing map:

$$\mathcal{M}(p,q) \times \mathcal{M}(q,r) \times (0,\varepsilon) \longrightarrow \mathcal{M}(p,r)$$

<u>Question of Arnol'd</u>

$H: M \times S^1 \longrightarrow \mathbb{R}$ on (M,ω) closed, symp.

$H_t: M \longrightarrow \mathbb{R} \rightsquigarrow X_H(t)$ Hamiltonian vector fields

$\{\phi_H^t\}_{t\in\mathbb{R}}$ s.t. $\phi_H^0 = Id$

$$\frac{d}{dt}\phi_H^t(p) = X_H(t, \phi_H^t(p))$$

The fixed points of the time-1 map ϕ_H^1

$\Leftrightarrow$ 1-periodic solution

$$\left\{\begin{array}{l} x: S^1 \longrightarrow M, \\ \dot{x}(t) = X_H(t, x(t)) \\ [x] = 0 \end{array}\right\} := \mathcal{P}(H)$$

<u>Arnol'd Conjecture</u>:

$$\# \mathcal{P}(H) \geq \sum_i b_i \qquad \text{Betti numbers.}$$

Poincaré - Hopf $\Rightarrow$ # crit pts $\geq \sum (-1)^i b_i$

Morse $\Rightarrow$ # crit pts $\geq \sum b_i$

Lefschetz $\Rightarrow$ # fix pts $\geq \sum (-1)^i b_i$

Floer $\Rightarrow$ # fix pts $\geq \sum b_i$

Need to assume <u>monotonicity</u>:

$$\int_{S^2} v^* c_1 = \tau \int_{S^2} v^* \omega \quad \text{for some } \tau > 0.$$

Consider $\mathcal{L}M$ - contractible loop space

$$= \{ x : \mathbb{R} \to M \; ; \; x(t+1) = x(t) \;\; \forall\, t \in \mathbb{R} \}.$$

$\xi \in T_x \mathcal{L}M \iff \xi : \mathbb{R} \to TM, \; \xi(t) \in T_{x(t)} M$

$$\xi(t+1) = \xi(t) \quad \forall\, t \in \mathbb{R}$$

1-parameter $H \rightsquigarrow \Psi_H : T\mathcal{L}M \to \mathbb{R}$

$$\Psi_H(x, \xi) = \int_0^1 \omega(\dot{x}(t) - X(t, x(t)), \xi(t))\, dt$$

$a_H : \mathcal{L}M \to \mathbb{R}/\mathbb{Z}$ whose diff. is Ψ_H.

$$a_H(x) = -\int_D u^*\omega - \int_0^1 H(t, x(t))\,dt \qquad u : D \to X$$

$$u|_{\partial D} \equiv x.$$

NB. a_H is $\mathbb{R}/\mathbb{Z}$-valued because $\int_{S^2} v^*\omega \in \mathbb{Z}$.

Gradient flow a_H: $\cdot\; J_t = J_{t+1} \in \mathcal{J}(M, \omega)$

$\cdot$ metric $\langle \xi, \eta \rangle_t = \omega(\xi, J_t \eta)$

$$\langle \xi, \eta \rangle = \int_0^1 \langle \xi(t), \eta(t) \rangle_t \, dt.$$

$$X_H = J \nabla H.$$

$\Rightarrow$ grad $a_H(x)(t) = J_t(x(t))\dot{x}(t) - \nabla H(t, x(t))$

A gradient flow of a_H is a smooth 1-parameter family of ~~maps~~ loops

$$\mathbb{R} \to \mathcal{L}M$$

$$s \mapsto u(s, \cdot)$$

$$\frac{\partial u}{\partial s} + \text{grad } a_H(u(s, \cdot)) = 0$$

$$\Rightarrow \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H(t, u) = 0. \qquad (*)$$

s = const $\Rightarrow$ Ham. curve equation

t = const $\Rightarrow$ Morse flow equation

$H = 0$ $\Rightarrow$ J-hol. curve equation.

$$E(u) = \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X(t,u)\right|^2 \right) ds\, dt$$

<u>Prop</u>: If $u: \mathbb{R} \times \mathbb{R}/\mathbb{Z} \longrightarrow M$, ~~TFAE~~ satisfying $(*)$, TFAE

i) $E(u) < \infty$

ii) $x^{\pm} \in \mathcal{P}(H)$ $\displaystyle\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t)$

$\displaystyle\lim_{s \to \pm\infty} \partial_s u(s,t) = 0$

<u>Thm</u>: $\mathcal{H}_{reg} \subset C^{\infty}(M \times \mathbb{R}/\mathbb{Z})$

1-par. solutions are nice

~~Let~~ For $H \in \mathcal{H}_{reg}$,

$\mathcal{M}(x^-, x^+, H, J) \leftarrow (*)$ + asymptotic to $x^{\pm}$.

is finite dimension for all $x^{\pm} \in \mathcal{P}(H)$, $\forall\ H \in \mathcal{H}_{reg}$.

<u>Monotone</u>: $\exists\ \eta_H : \mathcal{P}(H) \longrightarrow \mathbb{R}$. $\forall\ u \in \mathcal{M}(x^-, x^+, M, J)$

$$\dim_u \mathcal{M}(x^-, x^+, H, J) = \mu(u, H)$$
$$= \eta_H(x^-) - \eta_H(x^+) + 2\tau E(u)$$

$x \in \mathcal{P}(H)$ $\quad u: D \to M$

$u|_{\partial D} = x$

- $\mu_H(x, A \# u) = \mu_H(x, u) - 2c_1(A)$
- $a_H(x, A \# u) = a_H(x, u) + \omega(A)$

$\Longrightarrow$ $\eta_H(x) = \mu_H(x;u) - 2\tau a_H(x;u)$
monotonicity
does not depend on the disk u.

$E(u) = a_H(x^-, u^-) - a_H(x^+, u^+)$ (where $u^+ := u^- \# u$)
$\mu(u,H) = \mu_H(x^-, u^-) - \mu_H(x^+, u^- \# u)$

minimal Chern number $N = \inf\{ r > 0 : \exists\ u : S^2 \to M, \int_{S^2} u^* c_1 = r \}$

$\bar\partial_{H,J} = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H(t,u)$ vec. field along u.

$u \in \mathcal{M}(x^-, x^+)$ $\quad \mathcal{X}_u \subset C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, u^* TM)$

all v.f. along u with exponential decay as $s \to \pm\infty$
Solutions of $(*)$ with finite energy are zero sets of
$\mathcal{F}_u : \mathcal{X}_u \to \mathcal{X}_u$

$\mathcal{F}_u = \Phi_u^{-1}(\xi)\, \bar\partial_{H,J}(\exp_u(\xi))$
parallel translation

linearisation of $\mathcal{F}_u$ is as written by Vera:
$D\xi = \partial_s \xi + J_0 \partial_t \xi + S\xi$

$D : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$

<u>Thm:</u> D Fredholm
$\operatorname{ind} D = \mu_{CZ}(\Psi^+) - \mu_{CZ}(\Psi^-)$.

$$
\begin{tikzcd}
d\Psi_t(x_0) : T_{x(s)} M \arrow[r] & T_{x(t)} M \\
\mathbb{R}^{2n} \arrow[u] \arrow[r, "\Psi_s(t)"'] & \mathbb{R}^{2n} \arrow[u]
\end{tikzcd}
$$

$$\mu_H(x,u) = n - \mu_{CZ}(\Psi_1).$$

For $H \in \mathcal{H}_{reg}$, D_u surjective $\forall x^\pm \in \mathcal{P}(H) \Rightarrow$

Prop: $\widehat{\mathcal{M}}^1(x^-, x^+, H, J) = \mathcal{M}^1(x^-, x^+, H, J)/\mathbb{R}$

$= \{u : \mu(u,H) = 1\}$

is finite for all $x^\pm \in \mathcal{P}(H)$.

Prop: (convergence modulo bubbling + breaking

$u^\nu \in \mathcal{M}(x^-, x^+)$, $\sup_\nu E(u^\nu) < \infty$

then $\exists$ finite set of pts $\{z_j\}$ and "broken" solution u, subsequence

$u^\nu \to u$ uniformly with all derivatives on

compact subsets of $\mathbb{R} \times S^1 \setminus \{z_j\}$.

~~Prop: $u^\nu \in \mathcal{M}(x^-, x^+, H, J)$~~

A "broken" solution is analogous to a broken Morse

flowline:

Defn: $CF_r(H) = \bigoplus_{\substack{x \in \mathcal{P}(H) \\ \mu_{CZ}(x,H) = r \ (\text{mod } 2N)}} \mathbb{F}\langle x \rangle$

$$\partial^F \langle y \rangle = \sum_{\substack{x \in \mathcal{P}(H) \\ \mu_{CZ}(x,H) = r-1 \\ (\text{mod } 2N)}} \sum_{[u] \in \mathcal{M}^1(y,x,H,J)} \varepsilon(u) \langle x \rangle$$

for $y \in \mathcal{P}(H)$, $\mu(y,H) = r \ (\text{mod } 2N)$

Thm: $\partial^2 = 0 \Rightarrow$ define $HF_r(H) := H_*(CF_r, \partial)$

Thm: Given two sets of input data (H^0, J^0), (H^1, J^1), there exists

$$P^{01} : HF_*(H^0, J^0) \longrightarrow HF_*(H^1, J^1)$$

such that

$$P^{12} \circ P^{01} = P^{02}.$$

Thm: There is an isomorphism

$$\phi^1 : HF_r(M, \omega, H^1, J^1) \longrightarrow \bigoplus_{\substack{j \equiv r \\ (\text{mod } 2N)}} H_j(M, \mathbb{F})$$

which is functorial.