~~Holomorphic~~ Compactness for Holomorphic Curves in Cylindrical Manifolds

Tim

Target: $(\mathbb{R} \times V, J)$

J cylindrical: $\cdot$ invariant under time translation

$\cdot\ J(\partial/\partial t) = R$ is horizontal, i.e. $R \subset TV$

J determined by: $\xi = J(TV) \cap TV$

$J_\xi = J|_\xi$

R

$\Rightarrow$ 1-form λ, $\lambda(R) = 1$

$\lambda|_\xi = 0$

λ symmetric, $\mathcal{L}_R \lambda = 0 \iff \imath_R d\lambda = 0$

ω maximal rank closed 2-form on V s.t.

$\cdot$ J is compatible with ω

$(\omega(J\cdot, \cdot)$ a metric$)$

adjusted to ω: $\mathcal{L}_R \omega = 0$

$\imath_R \omega = 0$.

E.g. (1) V is a contact mfd; λ contact form

(2) $\pi : V \to M$ principal S^1-bundle

λ is an S^1-connection 1-form

$TV = \ker \pi_* \oplus H$

$\cup\!\!\text{l}$

R

$\lambda(R) = 1$

Check: $\imath_R d\lambda = 0$

M symplectic w.r.t. $\overline{\omega}$ $\quad \omega = \pi^* \overline{\omega}$.

$F : (S, j) \longrightarrow (\mathbb{R} \times V, J)$

$F = (a, f)$ $a = \mathbb{R}$ component

$f : V$ "

$\pi : TV \longrightarrow \xi$ through R

$DF \circ j = J \circ DF \iff \pi \circ df \circ j = J \circ \pi \circ df$

$(f^* \lambda) \circ j = da$

Energy:

$$E_\omega(F) = \int_S f^* \omega$$

$$E_\lambda(F) = \sup_{\varphi \in C} \int_S (\phi \circ a)\, da \wedge f^* \lambda$$

where C = non-neg. comp. supp. $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$

$\int_{\mathbb{R}} \phi = 1.$

$$E(F) := E_\omega(F) + E_\lambda(F).$$

__Lem:__ $E_\omega \geq 0$, $E_\lambda \geq 0$

If $E_\omega(F) = 0$ then its image $F(S)$ is tangent to R everywhere.

i.e. F is constant, or cylindrical over a Reeb orbit.

__Lem:__ If $F : D^2 \setminus 0 \longrightarrow \mathbb{R} \times V$

$E(F) < \infty$

If $\mathrm{im}(F) \subseteq K$ compact subset

F extends to smooth holomorphic $F : D^2 \to \mathbb{R} \times V$.

Prop: If R is Morse (or Morse-Bott)

Let $F = (a, f) : \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R} \times V$

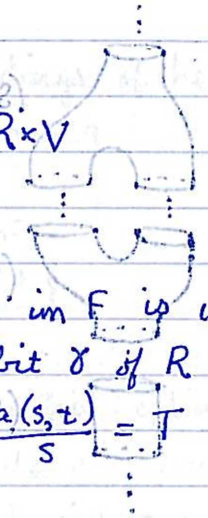
neg pos

$D^2 \setminus 0$

with $E(F) < \infty$ and suppose im F is unbounded

then $\exists\, T \neq 0$ and periodic orbit γ of R s.t.

$$\lim_{s \to \infty} f(s,t) = \gamma(Tt), \quad \lim_{s \to \infty} \frac{a(s,t)}{s} = T$$



Holomorphic buildings in cylindrical manifolds

(S, j, M, Z, D)

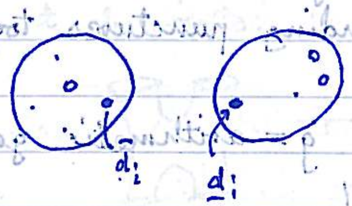
marked points, punctures, nodal points $= \{(\bar{d}_i, \underline{d}_i)\}_i$

Nodal holomorphic curve of height 1:

$F = (a, f) : (S \setminus Z, j) \longrightarrow (\mathbb{R} \times V, J)$

$F(\underline{d}_i) = F(\bar{d}_i)$

E.g.



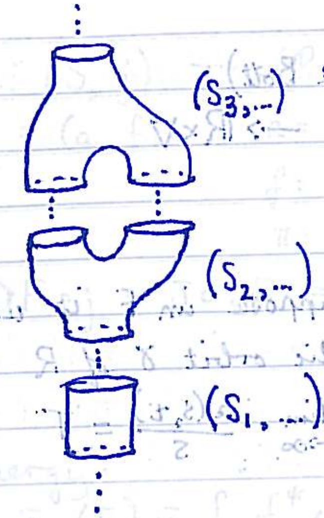
$\cdot$ = marked pt

$\circ$ = puncture

$\bullet$ = nodal pts.

height k holomorphic building:

$F_m = (S_m, j_m, M_m, Z_m, D_m) \quad m = 1, \ldots, k$



$Z_m = \overline{Z}_m \cup \underline{Z}_m$

pos. punctures neg. punctures

$\Downarrow$ blow-up

$\Gamma_m^+ \cup \Gamma_m^-$

$\Phi_m : \Gamma_m^+ \xrightarrow{\sim} \Gamma_{m+1}^-$ $m = 1, \dots, k-1$

(orientation-reversing)

$S_m^{Z_m} := S_m \setminus Z$ blown-up

$$S^{Z,\Phi} = S_1^{Z_1} \cup_{\Phi_1} S_2^{Z_2} \cup_{\Phi_2} \dots \cup_{\Phi_{k-1}} S_k^{Z_k}$$

$(\overline{F}, S^{Z,\Phi})$ is a map into V (you can glue Reeb orbits at corresponding punctures together).

To a building, associate g = arithmetic genus

$$g = \tfrac{1}{2}|D| - b_0 + \Sigma g_i + 1$$

conn. comp. top. genus

d = topological genus of resulting surface S^D obtained by gluing along blowup of D.

Defn: ${}^{K}\mathcal{M}_{g,\mu,p^-,p^+} = \{$ hol. buildings of height K with genus g, $|M| = \mu$, $|Z_1| = p^-$, $|Z_K| = p^+ \}$

$$\overline{\mathcal{M}}_{g,\mu} := \bigcup_{p^-,p^+ \geq 0} \bigcup_{K=1}^{\infty} {}^{K}\mathcal{M}_{g,\mu,p^-,p^+}$$

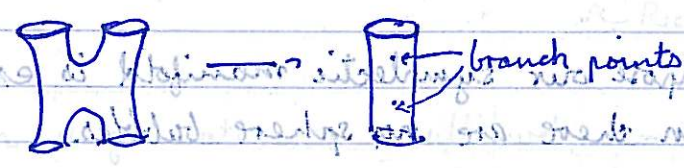
Thm: Let $(\mathbb{R} \times V, J, \omega)$ be symmetric cylindrical, J compact adjusted to ω. Then for any E,

$$\overline{\mathcal{M}}_{g,\mu} \cap \{\text{energy } E(u) \leq E\} \text{ is compact.}$$

A sequence of buildings (say height 1), what can happen:

(i) bubble at pt $\Rightarrow$ limiting building has 2 additional nodal pts ~~+ a new sphere~~

E.g. $V = S^1$, look at double cover of $\mathbb{CP}^1$ branched at 2 points:



Now suppose branch points move far apart - you get a holomorphic building of height 2: