Compactness of J-holomorphic spheres on a compact symplectic manifold

Sushmita

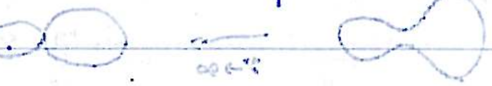
$\mathcal{M}(A, J) = \{ u : \Sigma \to M : \bar{\partial}_J u = 0 ,\ [u] = A \in H_2(M) \}$

is a fin. dim. mfd.

Is it compact?

$G = PSL(2, \mathbb{C})$ acts on $\mathbb{CP}^1$ - noncompact.

Look at $\mathcal{M}(A, J)/_G$.



Energy of $u : \Sigma \to M := \frac{1}{2} \int_\Sigma |du|^2 = \omega([u])$ if u is J-holomorphic

fixed $A \Rightarrow W^{1,2}$ bound on u

Elliptic regularity

$\bar{\partial}_J$ is elliptic $\quad \bar{\partial}_J u = 0$

If $u \in W^{k,p} \quad kp > 2$

$\Rightarrow u$ is smooth

$$\|u\|_{W^{k+1,p}} \leq C \left(\|u\|_{W^{k,p}} + \underbrace{\|\bar{\partial}_J u\|_{W^{k,p}}}_{=0 \text{ for hol. curve}} \right)$$

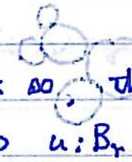
Prop: $u_\nu : \mathbb{CP}^1 \to M$, $\|u_\nu\|_{W^{1,p}} < C \quad p > 2$

$\Rightarrow$ there is a subsequence C^∞ converging on compact subsets.

$W^{k,p} \hookrightarrow C^{k-1}$ is compact $\quad p > 2$

Some results about J-hol curves

Removal of singularities.

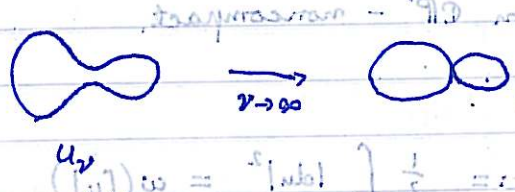


Given $u : B_r \setminus 0 \to M$, $E(u) < \infty$, then this extends to a smooth J-hol map $u : B_r \to M$.

Energy Quantisation

If $u : \mathbb{CP}^1 \to M$ non-constant, $E(u) \geq \hbar$ for some constant $\hbar$.

What can go wrong



Things go wrong if $\sup_\nu \|du_\nu\|_{L^\infty} = \infty$

Pick a sequence $\{z_\nu\} \subseteq \mathbb{C}$ s.t. $|du_\nu(z_\nu)| \longrightarrow \infty$

$z_\nu \longrightarrow 0$ WLOG

$$c_\nu := |du_\nu(z_\nu)|$$

$$v_\nu(z) = u_\nu\left(z_\nu + \frac{z}{c_\nu}\right)$$

$$|dv_\nu(0)| = 1, \quad \|dv_\nu\|_{L^\infty} \leq 1$$

$\Rightarrow \quad v_\nu \xrightarrow{UCS} v \qquad$ (UCS = uniformly on compact subsets)

$v : \mathbb{C} \to M$

We have $0 < E(v) < \infty$

$\Rightarrow$ by removal of singularities, v extends to $v : \mathbb{CP}^1 \to M$.

So our J-holomorphic spheres can converge to reducible curves, such as

Gromov convergence

Defn: If $u_\nu : \mathbb{CP}^1 \to M$, it Gromov converges to a reducible curve $v = (v_1, \dots, v_N)$ if there exists a sequence of reparametrisations
$$\phi^j_\nu \in PSL(2, \mathbb{C}) \quad \text{s.t.}$$

1) $u_\nu \circ \phi^j_\nu \xrightarrow{\text{ucs}} v_j$

2) $\exists$ smooth reparametrisation f_ν (not holomorphic) s.t.
$$u_\nu \circ f_\nu \xrightarrow{C^0} V \quad (V: \mathbb{CP}^1 \to M \text{ is made up of all } v_j \text{ but not holomorphic})$$
(i.e. every point converges somewhere on V).

Gromov compactness theorem

Suppose $E(u_\nu) < C$

Then there is a subsequence that Gromov converges to a reducible curve.

Rmk: If $A \in H_2(M)$ can not be written as $A = A_1 + \dots + A_k$
A_i spherical, $\omega(A_i) > 0$,
then $\mathcal{M}(A, J)/_G$ is compact.

Proof of Gromov compactness

- A point $z \in \mathbb{C}$ is regular if for some $\varepsilon > 0$, $|du_\nu| < C$ on $B_\varepsilon(z)$.
- If not z is singular
- A singular point is rigid if it is singular in all subsequences

· Mass of a singular point

$$m(z) := \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E(u_\nu, B_\varepsilon(z)).$$

<u>Convergence modulo bubbling</u>

<u>Thm</u>: $E(u_\nu) < C$. Then $\exists$ subsequence with finitely many singular points $\{z_1, \dots, z_n\}$ and

$$u_\nu \xrightarrow{ucs} u \text{ on } \mathbb{CP}^1 \setminus \{z_1, \dots, z_n\}$$

<u>Lemma 1</u>: Any rigid singular point z has $m(z) \geq \hbar$.

<u>Pf</u>: Use Hofer's Lemma:

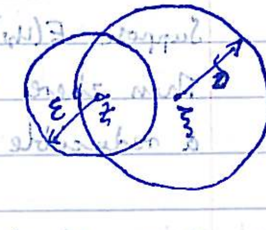
(If $f: X \to \mathbb{R}^+$ continuous

Given $B_\delta(\xi)$, $\xi \in X$, $\delta > 0$

then $\exists\, z \in B_\delta(\xi)$ and $\varepsilon < \delta$ s.t.

$$\sup_{B_\varepsilon(z)} f \leq 2 f(z)$$

$$\frac{\delta f(\xi)}{2} < \varepsilon f(z)$$



· Pick $\xi_\nu \to 0$ in $\mathbb{C}$ $|du_\nu(\xi_\nu)| \to \infty$

Pick $\delta_\nu \to 0$ $\delta_\nu |du_\nu(\xi_\nu)| \to \infty$

Hofer's $\Rightarrow \exists\, z_\nu \in B_{\delta_\nu}(\xi_\nu)$ $\varepsilon_\nu < \delta_\nu$

· Rescaling: $v_\nu(z) := u_\nu\left(z_\nu + \frac{z}{c_\nu}\right)$

$f = |du_\nu|$ $c_\nu := |du_\nu(z_\nu)|$

$\|dv_\nu\|_{L^\infty(B_{\epsilon_\nu c_\nu})} < 2$ Hard Rescaling

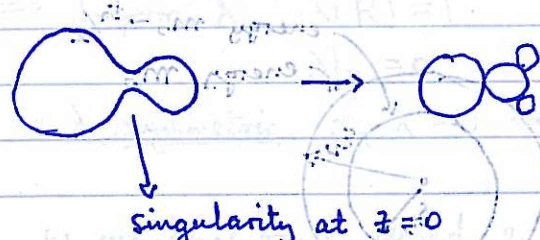
$\epsilon_\nu c_\nu \longrightarrow \infty$

S_σ : $v_\nu \longrightarrow v : \mathbb{CP}^1 \longrightarrow M$

$|dv(0)| = 1 \Rightarrow v$ nonconstant

$E(v) \geq \hbar$.

Soft Rescaling



$$\lim_{\epsilon \to 0} \lim_{\nu \to \infty} E(u_\nu, B_\epsilon) = m_0$$

Choose δ_ν s.t.

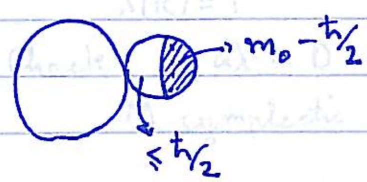
$$E(u_\nu, B_{\delta_\nu}) = m_0 - \hbar/2$$

$$\delta_\nu \longrightarrow 0$$

$$v_\nu(z) = u_\nu(\delta_\nu z)$$

$$\lim_{\nu \to \infty} E(v_\nu, B_1) = m_0 - \hbar/2$$

Need to prove $E(v, \mathbb{C} - B_1) = \hbar/2$



Another even softer rescaling:

Define ε_ν $\quad E(u_\nu, B_{\varepsilon_\nu}) = m_0 \qquad \delta_\nu < \varepsilon_\nu$

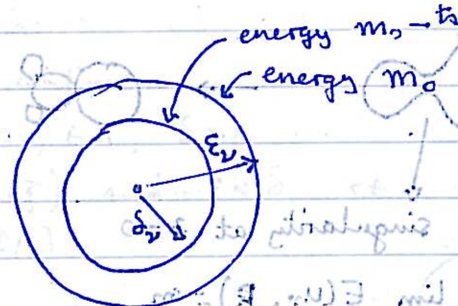
$W_\nu(z) := u_\nu(\varepsilon_\nu z) \longrightarrow$ this gives a const. curve

a). $\delta_\nu / \varepsilon_\nu \longrightarrow 0 \quad$ (on v_ν $E(v_\nu, B_{\varepsilon_\nu/\delta_\nu}) = m_0$

We know $\lim_{\nu \to \infty} E(v_\nu) \leq m_0 \Rightarrow \varepsilon_\nu/\delta_\nu$ unbounded

b) On W_ν, since $\delta_\nu/\varepsilon_\nu \longrightarrow 0$, all energy gets focused at 0.

We now have



$\delta_\nu / \varepsilon_\nu \longrightarrow 0$

<u>Annulus lemma</u>: $\exists\ c, \hbar, T_0 > 0$ s.t.

if $u : A(\varepsilon, R) \longrightarrow M$, J-hol.

$E(u) < \hbar$, then

$$E(u, A(e^T r, e^{-T} r)) < \frac{c}{T} E(u, A(r, R))$$

$\forall\ T \geq T_0$