~~Legendrian knots and other~~

Albin

Notation: $(\mathbb{R}^3, \alpha) = \left((q, p, u),\ du - p\,dq\right)$

$\mathcal{L} \subset \mathbb{R}^3$ Legendrian knot

$\Pi(q, p, u) = (q, p)$

Classical Invariants:

- Smooth isotopy type
- Maslov number $m(L) = \left| 2r(\Pi(\mathcal{L}))\right|$

  ↑ rotation number

- Thurston–Bennequin number

$$\beta(L) = lk(L, S(L))$$

↑ small shift of L in Reeb direction

$$= \sum_{\text{crossings}} \text{sign}(\partial_u, t_{over}, t_{under})$$

↑ Reeb v.f. ↑ tangent dir. to over strand ↑ " " under "

The dga (A, ∂) of a Legendrian knot:

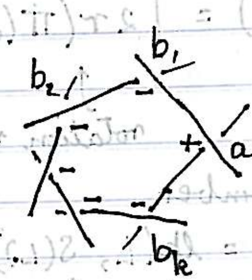
- non-commutative algebra over $\mathbb{Z}_2$ gen. by double points a_i.
- graded over $\mathbb{Z}/\mathbb{Z}\,m(L)$
- $\deg(a_i) = |a_i| = CZ(a_i) - 1 = 2\left(r(c) - \frac{1}{4}\right)$

where c is a path in $\Pi(L)$ from over point of α_i to under point.

Differential is defined by counting immersed disks with convex corners at double points. ($\Leftrightarrow$ rigid)

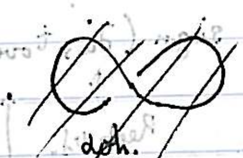
$$\partial a = \sum_{k \geq 0} \sum_{b_1, \ldots, b_k} |\mathcal{M}(a; b_1, \ldots, b_k)| \, b_1 \ldots b_k$$

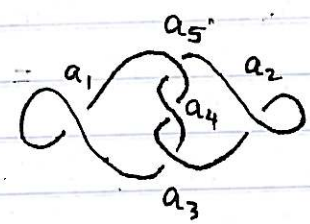
where $\mathcal{M}(a; b_1, \ldots, b_k)$ counts disks



This differential has degree -1.

E.g. Trefoil.





$$|a_1| = |a_2| = 1$$
$$|a_3| = |a_4| = |a_5| = 0$$

$$\partial a_1 = 1 + a_3 + a_5 + a_3 a_4 a_5$$
$$\partial a_2 = 1 + a_3 + a_5 + a_5 a_4 a_3$$
$$\partial a_3 = \partial a_4 = \partial a_5 = 0.$$

Stabilisation

Given a semi-free dga (A, ∂) graded by $\mathbb{Z}/m\mathbb{Z}$, where $A = T(a_1, \ldots, a_n)$, the ith stabilisation of (A, ∂) is the dga

$$S_i(A, \partial) = \left(T(a_1, \ldots, a_n, e_1, e_2), \partial'\right)$$

where $\deg(e_1) = i$

$\deg(e_2) = i-1$

$\partial'(a_j) = \partial a_j$

$\partial'(e_1) = e_2$

$\partial'(e_2) = 0.$

Tame isomorphism

An automorphism of $T(a_1, \ldots, a_n)$ is elementary if

$\varphi(a_i) = a_i \quad \forall\, i \neq j$

$\varphi(a_j) = a_j + u$

where $u \in T(a_i,\ i \neq j)$.

An isomorphism is called tame if it is the composition of elementary isomorphisms with a map $T(a_1, \ldots, a_n) \to T(b_1, \ldots, b_n)$.

Linearised homology

$$T(a_1, \ldots, a_n) = \bigoplus_{l=0}^{\infty} A_l$$

where A_l is spanned by words of length l. Denote

$$\partial_l = \Pi_{A_l} \circ \partial.$$

If $\partial_0 = 0$ then $\partial_1^2 = 0$, $\partial_1(A_1) \subset A_1$

$\Rightarrow$ can look at $H_*(A_1, \partial_1)$.

Exchange initial differential by new one s.t. $\partial_0 = 0$.

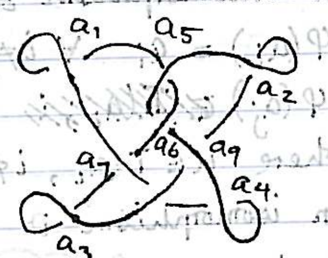
An augmentation is a graded algebra homomorphism $\varepsilon : A \longrightarrow A_0 = \mathbb{Z}_2$

$$g = g_\varepsilon : A \longrightarrow A$$
$$a \longmapsto a + \varepsilon(a)$$

Let G be the set of ~~augmentations~~ augmentations s.t. $\partial^g := g \partial g^{-1}$ satisfies $\partial_0^g = 0$.

The set of isomorphism classes of
$$\{ H(A, \partial_1^g) \,,\; g \in G \}$$
is an invariant for Legendrian isotopy.

E.g. A



$|a_i| = 1$ for $i \leq 4$

$|a_5| = 2$

$|a_6| = -2$

$|a_i| = 0 \quad i = 7, 8, 9$

$$\partial(a_1) = 1 + a_7 + a_7 a_6 a_5$$
$$\partial(a_2) = 1 + a_9 + a_5 a_6 a_7$$
$$\partial(a_3) = 1 + a_8 a_7$$
$$\partial(a_4) = 1 + a_8 a_9$$
$$\partial(a_5) = \dots = \partial(a_9) = 0.$$

We want $g(a_i) = a_i + c_i$ $(c_i \in \mathbb{Z}_2)$ s.t. $(g\partial g^{-1})_0 = 0$

$$\Longleftrightarrow \quad (g\partial)_0 = 0 \qquad (\text{since } \partial c_i = 0)$$

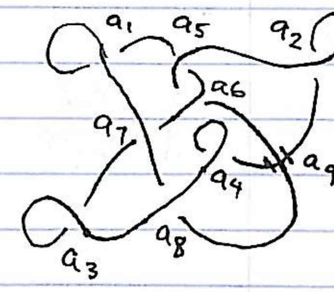
$$\Longleftrightarrow \quad \begin{cases} 1 + c_7 = 0 \\ 1 + c_9 = 0 \\ c_i = 0 \quad i \neq 7,8,9 \quad (\text{grading}) \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} c_7 = c_9 = 1 \\ c_8 = \text{arbitrary} \\ c_i = 0 \quad \text{else} \end{cases}$$

$$\partial_1^g(a_1) = (g\partial a_1)_1 = (1 + (a_7 + c_7) + (a_7 + c_7)a_6 a_5)_1$$
$$= a_7$$
$$\partial_1^g(a_2) = a_9$$
$$\partial_1^g(a_3) = a_8 + a_7$$
$$\partial_1^g(a_4) = a_8 + a_9$$
$$\partial_1^g(a_i) = 0 \quad \forall\, i \geq 5.$$

$$\Rightarrow \quad I(A, \partial) = \{H_2 = \mathbb{Z}_2,\ H_1 = \mathbb{Z}_2,\ H_{-2} = \mathbb{Z}_2\}$$

~~In contrast~~ In contrast, the knot B:

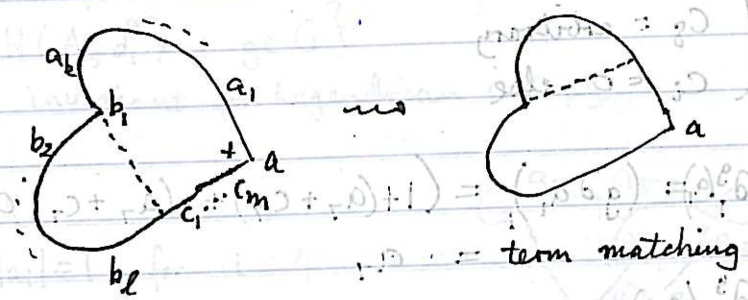


has the same classical invariants, but no generator of degree -2 $\Rightarrow$ can't have same Legendrian contact homology as A

$\Rightarrow$ A, B are distinguished by LCH, but <u>not</u> classical invariants.

<u>$\partial^2 = 0$:</u>

Leibniz rule $\Rightarrow$ suffices to check on generators.
Let $a_1 \dots a_k b_2 \dots b_\ell c_1 \dots c_m$ be a term in $\partial^2 a$, where $a_1 \dots a_k b_1 c_1 \dots c_m$ is a term in ∂a and $b_2 \dots b_\ell$ is a term in ∂b_1



: term matching with opp. sign

The holomorphic curve argument would look like