Algebraic Structures on SH

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Outline: I. TQFT structure (product, unit)

II. BV operator

III. More about BV algebras.

I. Ref: Ritter's paper.

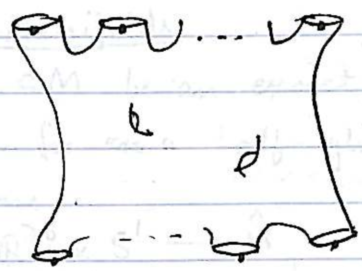
generators of SH^* are parametrised orbits of a t-indep't Hamiltonian, for our purposes.

Differential counts maps

$|x_-| - |x_+| = 1$

modulo translation

Why not count moduli spaces



want this to give map $(SH^*)^{\otimes I} \longrightarrow (SH^*)^{\otimes J}$

fixed Σ, asymptotic markers, H linear on the symplectisation part of boundary.

Fix weights $h_1, \ldots, h_I$, $k_1, \ldots, k_J$ > 0

On Σ, fix cylindrical ends $\simeq S^1 \times [0,\infty)$ for pos. ends
$S^1 \times (-\infty, 0]$ for neg. ends
($1 \in S^1$ = asymptotic marker)

Fix 1-form β on Σ s.t.
(i) $d\beta \leq 0$
(ii) pull back of β to cylindrical end of weight w is $w\,dt$.

By Stokes' theorem
$$0 \geq \int_\Sigma d\beta = \sum h_i - \sum k_j$$

In particular there's at least one outgoing end.
Solve $(du - X \otimes \beta)^{0,1} = 0$ (*) with usual asymptotics
Claim: pullback of (*) with weight w gives
$$\partial_s u + J(\partial_t u - wX) = 0$$

$\Rightarrow$ get maps
$$\begin{array}{c} HF^*(h_1 H) \otimes \cdots \otimes HF^*(h_I H) \\ \big\downarrow \Phi_{\Sigma, I, J, h_1 \ldots h_I, k_1 \ldots k_J} \\ HF^*(k_1 H) \otimes \cdots \otimes HF^*(k_J H) \end{array}$$
this is natural w.r.t. continuation
$\Rightarrow$ get $SH^{\otimes I} \xrightarrow[\Phi_{\Sigma, I, J}]{} SH^{\otimes J}$ $\qquad J \geq 1$.

Claim: these maps glue nicely: let $\Sigma_1 \# \Sigma_2 := \Sigma_1$ glued to Σ_2 with parameter λ

 Σ_1

 Σ_2

$$\Phi_{\Sigma_2} \circ \Phi_{\Sigma_1} = \Phi_{\Sigma_1 \#_\lambda \Sigma_2}$$

This gives

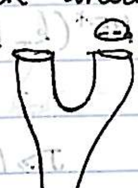
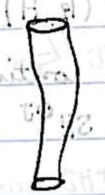
a) Ring structure

 product

b) Unit  $k \longrightarrow SH^*(\hat{M})$

$1 \longmapsto e$

check unitality:

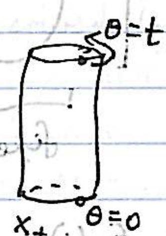
 =  = identity

in fact these are all constants disks $\Rightarrow$ unit corresponds to minimum of H, or

$H^*(\hat{M}) \longrightarrow SH^*(\hat{M})$

$1 \longmapsto e$

Consider the 1-parameter family of moduli spaces

$$\mathcal{M}_{BV}(x^-, x^+) = \bigcup_t$$ 

$$\dim \mathcal{M}_{BV}(x^-, x^+) = 1 + \dim \mathcal{M}(x^-, x^+)$$

this is rigid if $|x_+| - |x_-| = -1$.

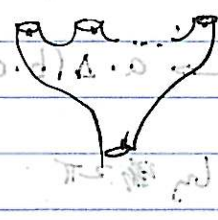
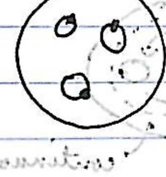
Get an operator $\Delta : SH^*(\hat{M}) \longrightarrow SH^{*-1}(\hat{M})$
this is the __BV operator__.

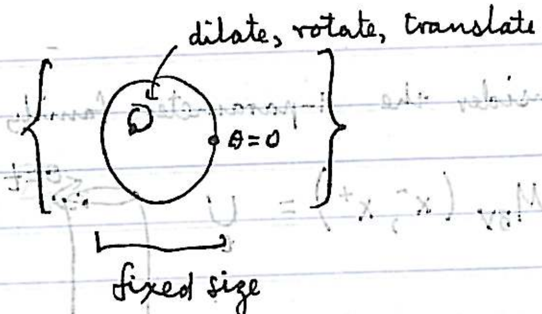
__Defn:__ A __BV algebra__ is a graded commutative (dg) algebra $(V, *)$ with a map
$$\Delta : V^* \longrightarrow V^{*-1}$$
s.t.

(i) $\Delta^2 = 0$

(ii) $$\Delta(xyz) = \Delta(xy)z + (-1)^{|x|} x \Delta(yz) + (-1)^{(|x|+1)|y|} y \Delta(xz) - \Delta(x) yz - (-1)^{|x|} x \Delta(y) z - (-1)^{|x|+|y|} xy \Delta(z).$$

Why is Δ non-trivial?

__Rmk:__  $\cong$ 

Defn: $Fr(1) = \Big\{$  $\Big\}$

Each point in $Fr(1)$ gives an operator $SH(\hat{M}) \to SH(\hat{M}$

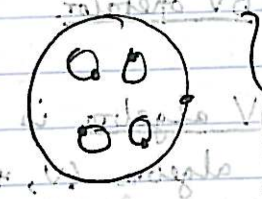
Point: $Fr(1)$ has non-trivial topology

$H_*(Fr(1)) = H_*(S^1)$

$S^1 \to Fr(1)$ representing $[S^1] \cdot 1$

gives a map

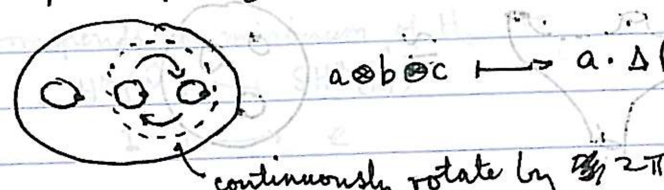
$SH^*(\hat{M}) \longrightarrow SH^{*-1}(\hat{M})$. Δ

In general, $Fr(k) = \Big\{$  $\Big\}$

operators $H_i(Fr(k)) \otimes SH^{\otimes k} \longrightarrow SH^{*-i}$

Say that SH^* is an algebra over H_* (framed little disks operad).

E.g.  gives a map $a \otimes b \otimes c \longmapsto a \cdot b \cdot c$.

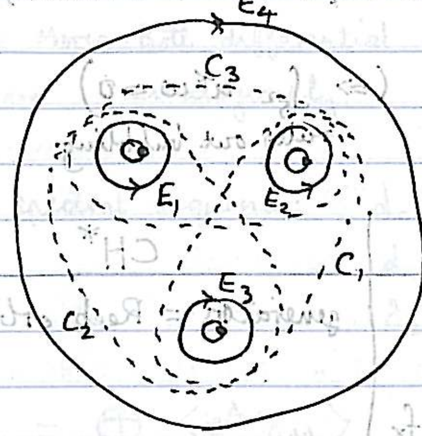


$a \otimes b \otimes c \longmapsto a \cdot \Delta(b \cdot c)$

Thm (Getzler):

All operators can be described by a combination of $*$, Δ.

BV axiom:



Lantern relation:

$$T_{E_4} = T_{E_1} T_{E_2} T_{E_3} T_{C_1} T_{C_2}$$