Here are several broad goals of the workshop we could keep in mind:

1) Understand equivalence between constructible sheaves and Fukaya category of cotangent bundle. Consider applications such as homological characterizations of compact exact branes, mirror symmetry for toric varieties, and Springer theory. In the first two cases, sheaves help us understand branes; in the third case, branes help us understand sheaves. All of the material here is available in the literature.

2) Understand what the above equivalence should imply about quantizations of more general exact symplectic targets arising in representation theory. For example, we should be able to see the relation between quantizations of Slodowy slices in the form of the Fukaya category and in the form of modules over W-algebras. To my knowledge, this is not completely mapped out by the literature.

3) Discuss directions for further investigation of relations between Fukaya categories and categories in representation theory. Starting point: Fukaya categories of cotangent bundles to flag varieties, $\mathcal{D}$-modules on moduli spaces of bundles. Here the literature points to many open questions.

1. Singularities and constructible sheaves.

1.1. **Tame geometry.** Subanalytic geometry. Defining functions. Whitney stratifications and triangulations. Thom isotopy lemmas. Example of real line.

Build up the notion of subanalytic subset of a real analytic manifold by starting with the real line and then considering standard operations (with an emphasis on the special role of the image of a map). Explain relation between closed subanalytic subsets and zeros of subanalytic functions. Discuss axioms of Whitney stratifications and results about stratifying and triangulating subanalytic sets. Discuss Thom isotopy lemmas, in particular the assertion: if $f : M \to N$ is a proper stratified map, then stratum-preserving homeomorphisms of N (smooth along each stratum) lift to stratum-preserving homeomorphisms of M (smooth along each stratum). Describe local structure of Whitney stratifications as iterated cone bundles along strata.

This lecture should contain many simple counterexamples. For example, to illustrate the Whitney conditions, one could discuss the Whitney umbrella and cusp.

Refs: [BM88], [VM96]

1.2. **Homotopical categories.** Differential graded and A_∞-categories. Functors and modules. Linear structure: shifts and cones. Localization with respect to collection of morphisms. Homological perturbation theory.

This lecture should approach categories as multi-pointed versions of algebras. In fact, we should have in mind the case where the number of points is finite so that in the end we could think in terms of algebras.

Introduce chain complexes and basic notions: tensor and hom, shift, sum and summand, cone, quasi-equivalence,... Introduce strong notion of algebra (differential graded algebra) and weak notion of algebra (A_∞-algebra). Draw operadic pictures for A_∞-categories and functors between them. Describe equivalence of differential graded categories and A_∞-categories via homological perturbation theory. Explain what is gained (and perhaps lost) in the stabilization (Morita theory) of A_∞-categories by passing to perfect modules: idempotent-completion of complexes of representable functors to chain complexes. Reminder that triangulated categories arise in nature as the underlying discrete categories of stable A_∞-categories. Describe localization of a category (basic example: passing from modules over an algebra to modules over a localization of the algebra).

Refs: [Ke06], [S], [L]

1.3. **Constructible sheaves.** Differential graded category of sheaves. Functoriality under maps. Standard triangles and bases. Relation to constructible functions.

This lecture can be in the more traditional language of triangulated categories as long as it is understood that all of the constructions and results can be lifted to the differential graded setting.

Fix Whitney stratification S of real analytic manifold X. Introduce differential graded category of S-constructible complexes. Discuss case where S consists of one stratum X itself (local systems and complexes with locally constant cohomology). Introduce Grothendieck's 6 operations (f^*, f_*), $(f_!, f^!)$, $\otimes$, $\mathcal{H}om$ and Verdier duality. Construct standard triangles associated to pair of an open $U \subset X$ and closed $V = X \setminus U$. Calculate morphisms between standard extensions of constant sheaves on strata of S. Possibly include: informal discussion of exit-path simplicial category of a Whitney stratification, and constructible sheaves as finitely-generated modules over the exit-path category. Explain how stalk Euler characteristic identifies Grothendieck group of constructible sheaves with constructible functions.

Refs: [KS84], [GM83]

1.4. **Examples.** Constructible sheaves on $\mathbb{R}$ stratified with a single marked point. Constructible sheaves on S^1 stratified with a single marked point. Constructible sheaves on $\mathbb{A}^1$ stratified with a single marked point. Constructible sheaves on $\mathbb{P}^1$ stratified with a single marked point.

The aim here is to give quiver presentations of categories of constructible sheaves in some simple examples. By choosing enough functionals, we can describe a category with the description depending on the functionals. For the above examples, choose various functionals and describe resulting quivers. Describe objects representing the functionals considered. For example, first construct quiver arising from considering generic stalk and stalk at marked point, then construct quiver using generic stalk and vanishing cycles at marked point.

Further example: a three stratum space such as $\mathbb{A}^2$ with a marked singular curve.

2. Microlocal geometry of sheaves

2.1. **Cotangent bundles.** Exact symplectic structure. Geodesic flow. Examples of Lagrangians: conormals, graphs and generalizations. Conormals to stratification. Lagrangian correspondences.

Summary of basic structures in exact symplectic geometry with emphasis on the case of cotangent bundles, including Liouville flow, contact hypersurfaces, compatible almost complex structures, exact Lagrangians,... Explain meaning of basic objects in terms of classical mechanics. Describe graph Lagrangians and conormal Lagrangians and their hybrids. Construct Lagrangian correspondences of cotangent bundles from maps of base manifolds, emphasizing case of projection and inclusion.

Refs: [A], [KS84]

2.2. **Characteristic cycles.** From constructible sheaves to conical Lagrangian cycles. Functoriality under maps.

Introduce group of conical Lagrangian cycles. Construct characteristic cycle of constructible sheaf on a manifold. Calculate everything in case when manifold is real line or complex line. Explain functoriality for Grothendieck's 6 operations (f^*, f_*), $(f_!, f^!)$, $\otimes, \mathcal{H}om$ and Verdier duality. Show characteristic cycle construction descends to isomorphism between group of constructible functions and group of conical Lagrangian cycles.

Refs: [KS84], [SV96]

2.3. **Intersection of Lagrangian cycles.** Perturbations near infinity. Intersections of characteristic cycles: compatibility with ext-pairing of constructible sheaves and corresponding pairing of constructible functions. Index theorems.

Describe framework of perturbing conical Lagrangians by normalized geodesic flow near infinity. Discuss $\mathbb{Z}/2$-grading on intersections of conical Lagrangian cycles. Show characteristic cycle takes pairing on constructible functions to intersection of conical Lagrangian cycles. Dubson-Kashiwara index formula (generalization of Poincaré-Hopf index formula): calculate global Euler characteristic of constructible sheaf as intersection with zero section.

Construct automorphisms of group of conical Lagrangian cycles via motions of pieces of support. Example of Dehn twist on conical Lagrangian cycles in T^*S^1. This topic is logically independent of the preceding but is reasonable to discuss at this juncture.

Refs: [GrM97], [NZ09]

2.4. **Riemann-Hilbert correspondence.** Differential operators as quantization of functions on cotangent bundle. Algebraic model of constructible sheaves: regular holonomic $\mathcal{D}$-modules.

Explain Riemann-Hilbert correspondence between regular holonomic $\mathcal{D}$-modules and constructible sheaves. Discuss the failure of an abelian version and the resulting notion of a perverse sheaf. Introduce the singular support of a $\mathcal{D}$-module and its relation to characteristic cycles. Illustrate everything with the case of $\mathbb{A}^1$ stratified by a single marked point.

Refs: [Be], [Kap]

3. Exact Lagrangians in cotangent bundles

3.1. **Morse category of submanifolds.** Gradient tree A_∞-category of submanifolds with local systems. Equivalence with constructible sheaves.

This lecture should explain how the differential graded category of constructible sheaves on a manifold can be reformulated in terms of a Morse A_∞-category whose objects are locally closed submanifolds equipped with local systems.

Basic case: explain equivalence of de Rham algebra of compact manifold with Morse A_∞-algebra. This will provide opportunity to interpret A_∞-operad in terms of trivalent graphs. Show how Morse theory provides geometric ingredients to apply homological perturbation theory. (For bonus points: mention other sources of parallel geometric ingredients such as Hodge theory.) Main topic: interpret constructible sheaves in terms of Morse theory. Explain how Thom's isotopy lemma allows one to replace locally closed submanifolds with singular boundary with open submanifolds with smooth boundary. Draw vector fields for constructible sheaves and calculate morphisms, for example for $\mathbb{R}$ stratified with a single marked point, and $\mathbb{A}^1$ stratified with a single marked point. Possible further topic: explain some of Grothendieck's 6 operations in terms of Morse A_∞-category.

Refs: [HL01], [KS01], [NZ09]

3.2. **Exact Floer-Fukaya theory.** Fukaya category of compact exact Lagrangians in exact symplectic target. Brane structures. Moduli spaces of disks. Organization into A_∞-category.

This lecture should be an introduction to Fukaya categories of exact targets. Seidel's book provides the foundations and the speaker should choose appropriate highlights. It is likely more worthwhile that we understand the broad picture than the analytic details. We should know what brane structures are and why that is what they are (gradings of intersections and orientations of moduli spaces). We should hear enough about the behavior of moduli of disks to see the A_∞-structure (most prominently, there should be a discussion of the A_∞-equations coming from the boundary of moduli). We should hear enough about continuation maps to believe that everything is well-defined. The lecture can restrict to compact Lagrangians as we will be hearing about non-compact ones soon enough. Should discuss the Piunikhin-Salamon-Schwarz (PSS) calculation of endomorphisms of compact branes. Resolutions of du Val singularities and their deformations would be a good example to illustrate the theory (and will appear in later talks).

Refs: [S]

3.3. **Infinitesimal Fukaya category of cotangent bundle.** Noncompact branes: perturbations, tameness, bounds on disks. Comparisons with directed and wrapped Fukaya categories. Equivalence of subcategory of standard branes with Morse category of submanifolds.

This talk should consist of roughly two halves: general theory of Fukaya categories with non-compact branes and example of the cotangent bundle.

First half. Survey general techniques for dealing with disks along noncompact branes: energy bounds, tameness, diameter estimates. For exact target with fixed energy function, introduce the infinitesimal Fukaya category where small Hamiltonian perturbations of branes are used near infinity. Comparisons could be made with directed Fukaya-Seidel categories of Lefschetz fibrations, and also wrapped Fukaya categories where the Hamiltonian perturbations are not small but rather linear near infinity.

Second half. Explain why the Morse A_∞-version of constructible sheaves embeds in the infinitesimal Fukaya category of the cotangent bundle. Here the main ingredient is Fukaya-Oh's analytic equivalence between gradient trees and pseudo-holomorphic disks (or alternatively, hybrid moduli spaces interpolating between them).

Refs: [S], [Sik94], [FO97], [NZ09], [Nspr]

3.4. **Equivalence of sheaves and branes.** Formalism of Yoneda lemma and bimodules. Beilinson's argument. Decomposition of diagonal. Noncharacteristic motions.

The aim of this talk is to prove that the infinitesimal Fukaya category of the cotangent bundle is equivalent to constructible sheaves. At this point, what is left to prove is that the standard branes coming from standard sheaves indeed generate.

Begin with general discussion of the Yoneda lemma, functors and bimodules, and the formalism of generators. Introduce Beilinson's construction of generators for coherent sheaves on projective space as guiding example. Bulk of talk should be devoted to applying this argument to the infinitesimal Fukaya category of the cotangent bundle. Here the main ingredient is the notion of non-characteristic propagation. Thom's isotopy lemma should be reinterpreted in the language of non-characteristic maps. An analogous lemma for continuation maps of branes should be formulated. Finally, we should see at least a sketch of Beilinson's argument in the setting of the infinitesimal Fukaya category. Application: homological characterization of compact exact branes in cotangent bundle.

Refs: [B78], [N09], [Nspr]

4. Some examples and applications

This day's talks are more independent of each other and the specific material covered can be determined by the speaker's taste.

4.1. **Mirror symmetry for toric varieties.** Fukaya category of cotangent bundle of torus.

Consider torus $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$. Introduce alternative viewpoints on symplectic geometry of $(\mathbb{C}^\times)^n \simeq T^*(S^1)^n$ via two projections $T^*(S^1)^n \to (S^1)^n$ and $T^*(S^1)^n \to (\mathbb{R}^\vee)^n$. Describe branes arising from considering a toric compactification of $(\mathbb{C}^\times)^n$. Explain how to think about them in terms of constructible sheaves. Discuss mirror symmetry and dual description of coherent sheaves in terms of constructible sheaves. Extra credit: equivariant generalization.

If we understand nothing else, we should at least understand mirror symmetry between A-model of T^*S^1 and B-model of $\mathbb{P}^1$.

Refs: [FLTZ] and related papers.

4.2. **Springer theory.** Fukaya category of cotangent bundle of Lie algebra. Fourier transform from Floer perspective.

Describe basic diagram of Springer theory arriving at Springer brane in $T^*\mathfrak{g}$. Introduce Fourier dual perspective and Fourier transform for branes. Deduce consequences for Springer brane. Interpret preceding in classical language of constructible sheaves.

Refs: [BoM81], [Nspr]

4.3. **Microlocalization and Hamiltonian reduction.** Formalism of microlocalization and Hamiltonian reduction. Introduction to crepant resolutions, their deformations and quantizations.

This talk and the one that follows could be planned in tandem. One approach would be to have the first talk cover theory, and the second cover examples. In any event, the speakers should strategize together.

We should see that many important examples of exact symplectic manifolds (symplectic resolutions with $\mathbb{C}^\times$-action) arising in representation theory can be constructed from conical open subsets of cotangent bundles via Hamiltonian reduction. We should learn how to think about categories (Fukaya, modules over deformation quantization) associated to such targets and their deformations can be arrived at from categories (Fukaya, microlocal constructible sheaves and $\mathcal{D}$-modules) associated to conical open subsets of cotangent bundles via Hamiltonian reductions.

4.4. **W-algebras from topological viewpoint.** Fukaya category beyond compact branes in Slodowy slices.

This talk could map out the relation between branes in Slodowy slices and their deformations, sheaves on flag manifolds (or equivalently, regular holonomic $\mathcal{D}$-modules on flag manifolds), and modules over W-algebras. The specific example of du Val resolutions could be discussed concretely.

Refs: [KhS], [SS], [Ma], [Lo] among many related papers.

5. Further directions

5.1. **Gauge theory setting.** Hitchin integrable system. Relation to talks of previous day. Challenge of quantization of fibers.

Refs: [BD], [KW], [Kap]

5.2. **Where to go from here.**

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