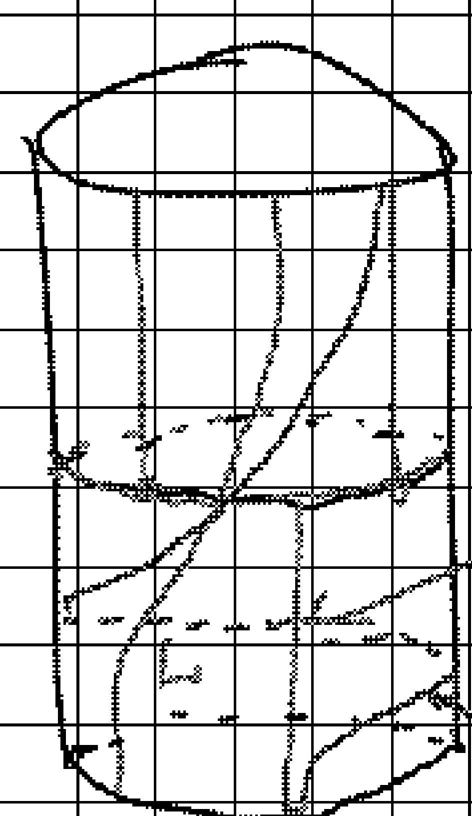
Wrapped category:

perfect duality   To be done

$\mathrm{Fuk}_\Lambda(T^*X) \simeq D^b_{\mathrm{cone}(\Lambda) + \text{zero section}}(X)$ | $\mathrm{WFuk}_\Lambda(T^*X) \sim D^b_{\mathrm{cone}(\Lambda) + \text{zero section}}(X)$

This week | (done by Mohammed in case Λ empty)

$\mathrm{WFuk}_\Lambda(T^*X)$ = Lagrangians avoiding Λ at ∞ (?) so that wrapping ~~does~~ flow dies near Λ (partially wrapped)

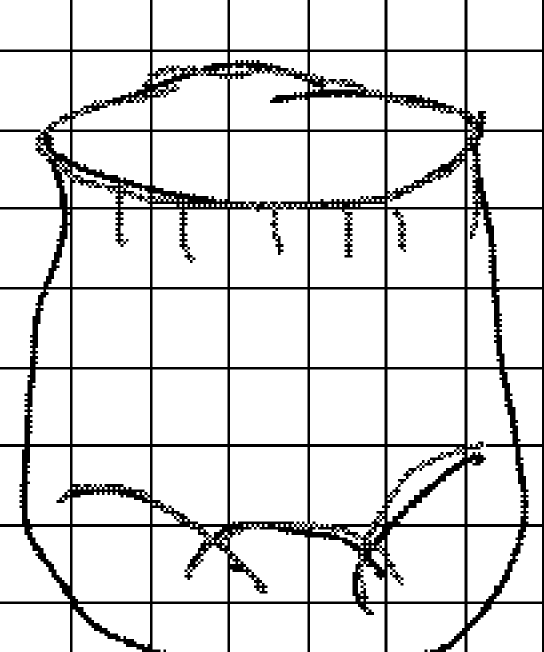


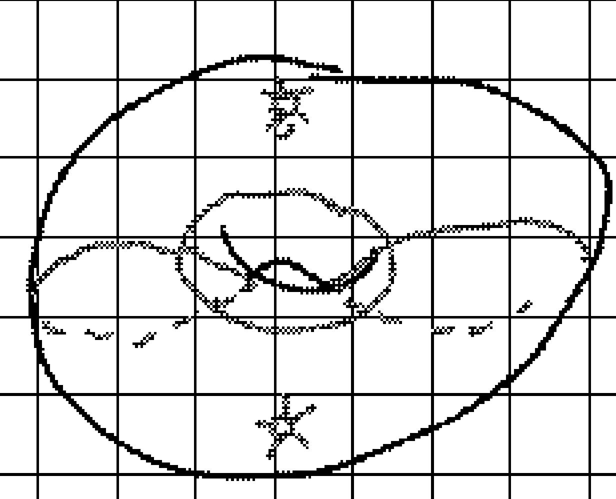
wrapping of L - can't go past Λ.

Gauge Theory:

Hol. symp. vs symp.

Ex: $\tilde{S}_e$ = complex symp surface | $\mathrm{Re}(\tilde{S}_e)$ = Riemann surface





$\mathrm{Bun}_G(\Sigma)$ – Σ a Riem. Surf., $G/\mathbb{C}$ reductive.
Kähler

2) $T^*\mathrm{Bun}_G(\Sigma)$ – hyperkähler
NB $\mathrm{Bun}_G(\Sigma) \subset T^*\mathrm{Bun}_G(\Sigma)$ Lagrangian $\Rightarrow$ symp. forms not compatible

Geom. Langlands

Donaldson theory
($=$ S-W theory)

4d TFT:

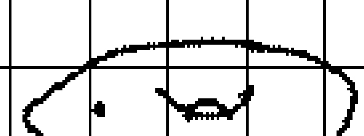
4-mfd $\rightsquigarrow$ number
3-mfd $\rightsquigarrow$ vec. space
2-mfd $\rightsquigarrow$ category
$\vdots$
pt 3-category

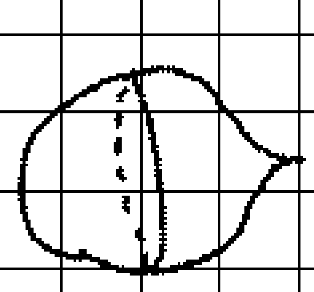
trade off geometric vs. algebraic complexity.

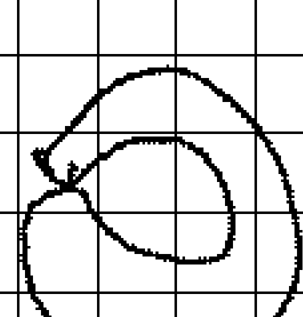
$\rightsquigarrow$ study Fukaya category of $T^*\mathrm{Bun}_G(\Sigma)$.
(very important category).

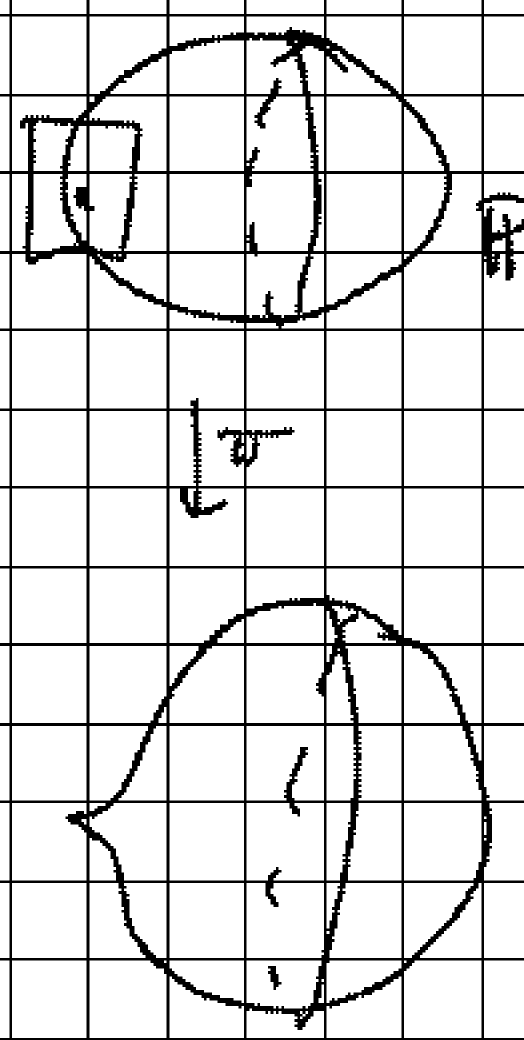
Geometric Langlands Gauge Theory for Σ of small genus:

Which curves are group schemes?

1)  E

2)  cuspidal ell. curve "$\mathbb{A}^1$"

3)  nodal ell. curve "$\mathbb{G}_m$".



$$\begin{array}{c} \mathrm{Bun}_G(\mathbb{P}^1) \\ \uparrow \pi^* \\ \mathrm{Bun}_G(C) \end{array}$$

1) $\mathrm{Bun}_G(\mathbb{P}^1)$ (let $G = GL_n$ for convenience)

Groth. thm $\Rightarrow$ $V^n = \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_n)$
$\quad\quad\quad\quad\quad\quad\quad\quad |$
$\quad\quad\quad\quad\quad\quad\quad\quad \mathbb{P}^1$

$\Lambda \simeq \mathbb{Z}^n$

$W = \Sigma_n$.

$\mathrm{Bun}_G(\mathbb{P}^1) = L_0 G \backslash LG / L_\infty G$

loop group — loops extending over ∞.

should think of as a stack.

analogue of $B \backslash G / B$.

$\mathrm{Bun}_G^{ss}(\mathbb{P}^1) \overset{\text{open}}{\subset} \mathrm{Bun}_G(\mathbb{P}^1)$

$\quad ||$

moduli of triv bundles

$\quad ||$

$BG = pt/G$

$\mathrm{Bun}_G^{ss}(C) = \mathfrak{g}/G$ — vec. field at point $\in \mathbb{P}^1$ lying over cusp, which tells you how to descend to C.

$\mathrm{Bun}_G^{ss}(N) = G/G$

$\mathrm{Bun}_G^{ss}(E) =$ ell. version of group$/G$.

Now pass to T^*

$T^*\mathfrak{g}$, T^*G, $T^*(G_{ell})$

$\uparrow$

Springer theory.

$\hookrightarrow T^*\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g}^*$

$$\begin{array}{ccc} T^*(\mathfrak{g}/G) = \{(X,Y) \in \mathfrak{g}\times\mathfrak{g} \mid [X,Y]=0\}/G & & (X,Y) \\ \downarrow \mu & & \downarrow \\ \mathfrak{h}^*/W & & \text{eigenvalues of } Y. \end{array}$$

Why study $L_\lambda \subset T^*\mathfrak{g}$? It's a Springer brane.

$$\begin{array}{ccc} T^*\mathrm{Bun}_G(\Sigma) & \xrightarrow{\mu} & \mathbb{A} \\ \downarrow \pi & & \\ \mathrm{Bun}_G(\Sigma). & & \end{array}$$

big affine space

$$\left(\text{cf:} \quad \begin{array}{ccc} T^*S^1 & \longrightarrow & \mathbb{R} \\ \downarrow & & \\ S^1 & & \end{array}\right)$$

find eigenobjects of Dehn twists - for T^*S^1, it's the S^1-fibres (they're preserved by Dehn twists).

$\mathrm{Bun}_G(\Sigma) = \{\Sigma \to BG = pt/G\}$

$T^*\mathrm{Bun}_G(\Sigma) \underset{\substack{\text{almost} \\ \text{(up to twist)}}}{\sim} \{\Sigma \longrightarrow \mathfrak{g}^*/G\}$

$= \{(\mathcal{P}, \Phi) \mid \mathcal{P}_G \in \mathrm{Bun}_G$

$\Phi \in \mathfrak{g}^* \overset{G}{\times} \mathcal{P}_G\}$

Higgs field.

Define

$\mathbb{A} \underset{\text{almost}}{\sim} \{\Sigma \to \mathfrak{h}^*/W\}$

set of eigenvalues.

$$\begin{array}{ccc} x: \Sigma & \xrightarrow{x} & \mathfrak{g}^*/G \\ & \mu(x) \searrow & \downarrow \\ & & \mathfrak{h}^*/W \end{array}$$

$$\begin{array}{ccc} x & \in & T^*\mathrm{Bun}_G(\Sigma) \\ \downarrow & & \downarrow \mu \\ \mu(x) & \in & \mathbb{A} \end{array}$$

Hitchin fibration.

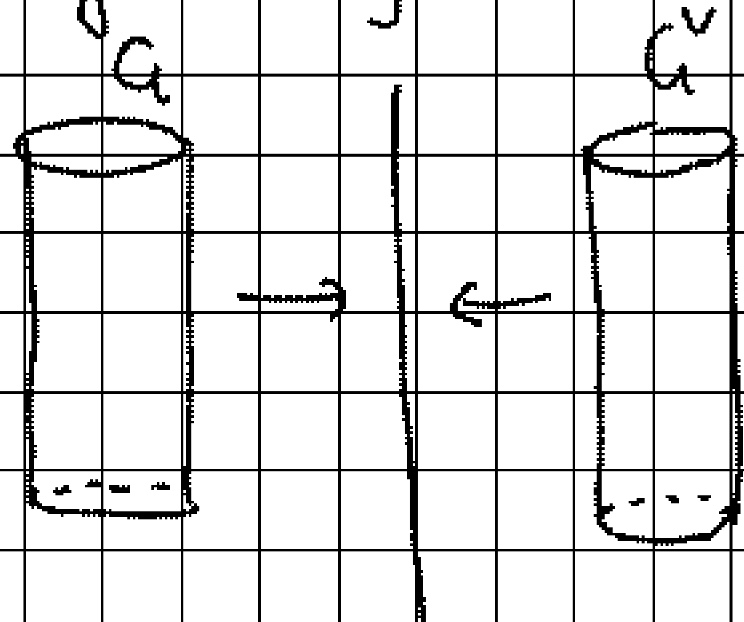
Want to understand

$$L_A \subset T^*\mathrm{Bun}_G(\Sigma) \qquad a \in A^{reg}.$$

Their Floer theory would be very interesting to repn theorists - who approach these from D-modules, constructible sheaves etc.

Mirror symmetry:

$G^\vee$ = Langlands dual



Hitchin fibrations are SYZ dual?