Chris Dodd - W-algebras

Note Title 6/16/2011

Hamiltonian Reduction

Quantization ⇝ D_X-modules (of T^*X)

↳ symplectic / Poisson variety with nice group action

↓ HR

new symp/Poiss variety.

Can Quantise X ⇒ can quantise $X /\!\!/ G$.

Examples: G - semisimple alg. gp. (e.g. $SL_n\mathbb{C}$)

$\mathfrak{g}^* \xrightarrow[\text{HR}]{\text{pick } e \in \mathcal{N}} S_e$ (poisson variety) (transverse slice to $G \cdot e$)

$\mathcal{U}(\mathfrak{g})$?

$gr(\mathcal{U}(\mathfrak{g}))$ $\xrightarrow{\text{HR}}$ $W(e)$

$\mathcal{O}(\mathfrak{g}^*) = sym(\mathfrak{g})$

$\mathfrak{g}^* \xrightarrow{\text{HR}} S_e$

$M \circlearrowright \mathfrak{g}^*$

$\mathfrak{g}^* \xrightarrow{\mu} \mathfrak{m}^*$, $\chi \in \mathfrak{m}^*$

HR a) $\mu^{-1}(\chi)$ as a variety

HR b) $M \circlearrowright \mu^{-1}(\chi)$; $S_e := \mu^{-1}(\chi) /\!\!/ M$

HR a) $\mathcal{O}(\mathfrak{m}^*) \xrightarrow{\mu^\vee} \mathcal{O}(\mathfrak{g}^*)$

$I_\chi \subseteq \mathcal{O}(\mathfrak{m}^*)$, $I_\chi \longrightarrow I_\chi \mathcal{O}(\mathfrak{g}^*)$ ← this defines $\mu^{-1}(\chi)$

$\mathcal{O}(\mu^{-1}(\chi)) = \mathcal{O}(\mathfrak{g}^*)/I_\chi$

HR b) $\mathcal{O}(S_e) = \left(\mathcal{O}(\mathfrak{g}^*)/I_\chi\right)^M$ ← M-invariants ⇕ quotient by M.

Now do the same thing to the $\mathcal{U}(\mathfrak{g}) \longrightarrow W_e$ row.

Quantum Ham. Red.

QHR a) $I_\chi \subseteq \mathcal{U}(\mathfrak{g})$; $\mathcal{U}(m) \to \mathcal{U}(\mathfrak{g})$

$$I_\chi := \langle m - \chi(m) \mid m \in \mathfrak{M} \rangle$$ (different from previous I_χ).

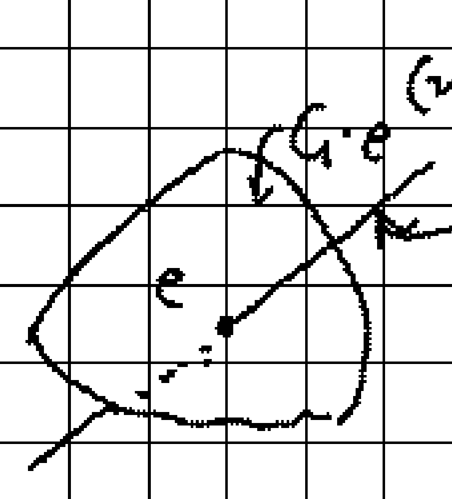
$\mathcal{U}(\mathfrak{g})/I_\chi$ (NB. not a ring: I_χ is only a left ideal)

QHR b) $W(e) := \left(\mathcal{U}(\mathfrak{g})/I_\chi\right)^M$ this is an algebra (something to check here - cf doing a Lag. corresp, it's not an algebra if you only go halfway).

Thm: $W(e)$ is a filtered non-comm ring

$\mathrm{gr}(W(e)) \xrightarrow{\sim} \mathcal{O}(S_e)$.

Inspiration:



S_e = slice (actually affine)

$$T_e(\mathfrak{g}^*) = T_e(G\cdot e) \oplus T_e(S_e).$$

we have a Poisson structure on the whole thing, which restricts to a symp. struc. on $T_e(G\cdot e)$.

$$\mathcal{U}(\mathfrak{g})_{\hat{\chi}} \xrightarrow{\sim} \widehat{W}_d \mathbin{\hat{\otimes}} W(e)_{\hat{\chi}}.$$

If $M \in \mathrm{Mod}(\mathcal{U}(\mathfrak{g}))$

$\mathrm{Supp}(M) \subseteq \mathfrak{g}^*$

$\{M \in \mathrm{Mod}(\mathcal{U}(\mathfrak{g})) \mid \mathrm{Supp}(M) = \overline{G\cdot e}\}$ these are important in representation theory.

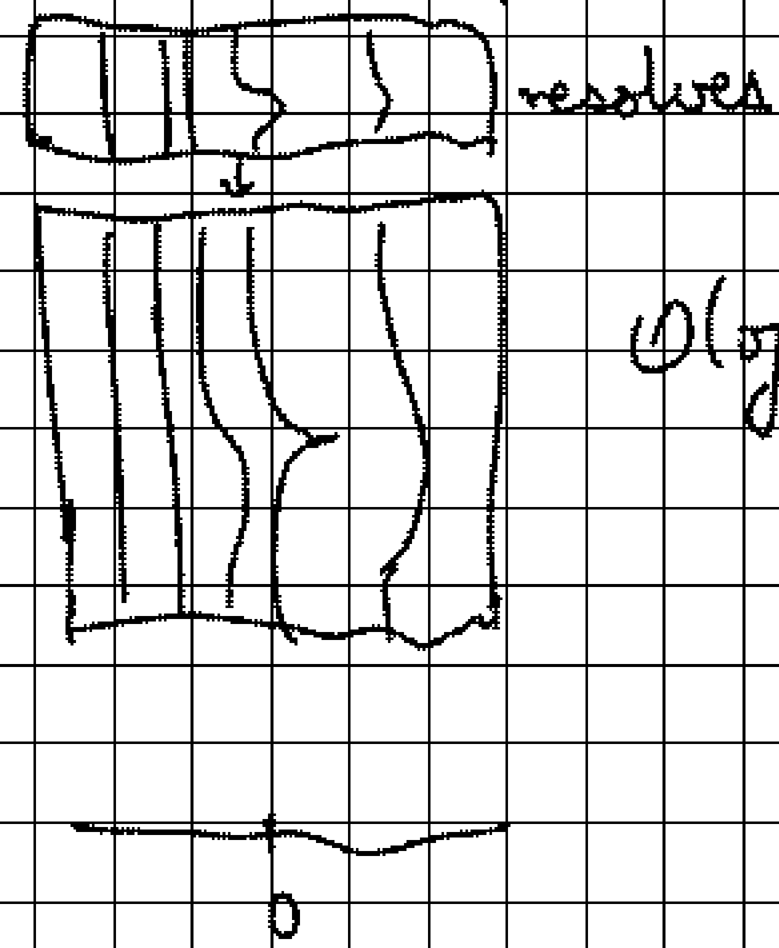
↑ closely related to fin. dim. $W(e)$-mod.

$\tilde{S}_e^\circ \xleftarrow{HR} T^*(G/B) \rightsquigarrow D_{G/B} \xrightarrow{QHR} \tilde{W}(e)_0$ quantises $\tilde{S}_e^\circ$

$\downarrow \qquad \downarrow \rho \qquad \downarrow \rho_*(D_{G/B}) \cong U(\mathfrak{g})_0 \qquad \downarrow$

$S_e^\circ \xleftarrow{HR} \mathcal{N} \rightsquigarrow U(\mathfrak{g})_0 = U(\mathfrak{g})/Z_\chi \xrightarrow{QHR} W(e)_0$

Picture:



$\mathcal{O}(\mathfrak{g}^*) = Sym(\mathfrak{g})$

quantise

$U(\mathfrak{g})$

$\uparrow$

$\mathcal{O}(\mathfrak{h}^*/\!/W) \cong Z_{\mathfrak{g}}$

Vague generalities: $Rep(W(e)_0) \xrightarrow[\sim]{fact} Mod(\tilde{W}(e)_0)$

hope

$Fuk(\tilde{S})$ ↖ hope: Fuk HR (Lag corresp).

$Rep(U(\mathfrak{g})_0) \xrightarrow{\sim} Mod(D_{G/B}) \sim Fuk(T^*G/B)$

Algebraic version of getting from $W(e)$-mod to $U(\mathfrak{g})$-mod.
(HR on modules)

$\widetilde{\mu^{-1}(\chi)} \longrightarrow \tilde{S} \qquad [\mu^{-1}(\chi)/\!/M = S]$

↑ principal M-bundle

(pull everything back to resolution)

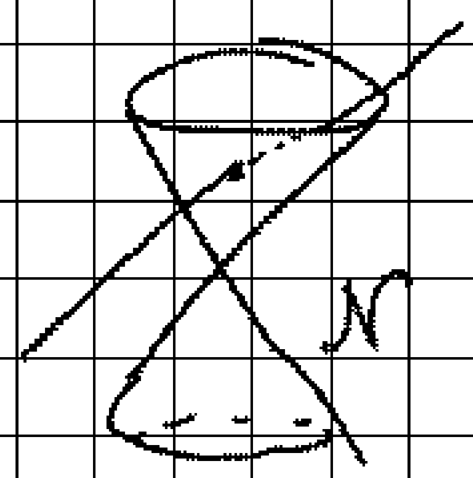
$Sh(\tilde{S}) = Sh(\widetilde{\mu^{-1}(\chi)}; M)$ ↖ M equivariant

$Mod(\tilde{W}(e)_0)$ $\qquad D_{G/B}$-mod; (M, χ).

$\searrow \sim$ // everything here is supported on $\widetilde{\mu^{-1}(\chi)}$.

$Mod(U(\mathfrak{g})_0; (M, \chi))$

E.g. $G = SL_2$



$S_e = e + \lambda f$

$S_e^\circ = \{e\}$

$W(e)_0 = \mathbb{C}$

G/B for sl_2: $\{0 \subsetneq \ell \subsetneq \mathbb{C}^2\} \cong \mathbb{P}^1$

$M \supset N = \{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\}$

N acts on $\mathbb{P}^1$ by conjugation: orbits are $\{\infty\}$, $\mathbb{A}^1$.

induces iso. $N \xrightarrow{\sim} \mathbb{A}^1$.

Let $V \in (D_{G/B}\text{-mod}, N, \chi)$ module $V|_{\mathbb{A}^1}$

ex: on $\mathbb{A}^1$, $\left(D_{\mathbb{A}^1}/\partial - 1\right)$

on $\mathbb{P}^1$: $j_*(D_{\mathbb{A}^1}/\partial - 1)$