Travis Schedler - Slodowy slices

Note Title 6/16/2011

$$\begin{array}{ccc} \tilde{S}^\circ & \subseteq & T^*(G/B) \\ \downarrow & & \downarrow \\ S^\circ & \subseteq & \mathcal{N} \end{array}$$

see Nick R.'s talk

$\mathcal{N}$ ← admits $\mathbb{C}^*$ dilation

We'll define $\tilde{S}^\circ$, and an object W_e°, a n.c. algebra, such that $W_e^\circ\text{-mod} \underset{D^b}{\simeq} \mathrm{Fuk}(\tilde{S}^\circ)$.

Some nice properties of $\downarrow$

1) symp. resoln. = proper, birational, Poisson

- comm. alg.
- $\{,\}$ Lie br.
- $\{fg, h\} = f\{g,h\} + g\{f,h\}$

E.g. (X, ω) affine symp var. ⇒ smooth Poisson.

$$\omega^{-1}(df)(g) =: \{f, g\}.$$

E.g. $\mathfrak{g}^*$, where $\mathfrak{g}$ = f.d. Lie alg

$\mathcal{O}_{\mathfrak{g}^*} = \mathrm{Sym}\,\mathfrak{g}$

$[,]_{\mathfrak{g}}$ extends uniquely to $\{,\}$ on $\mathcal{O}_{\mathfrak{g}^*}$

$\mathcal{N} \subseteq \mathfrak{g}^*$ is a closed Poisson subvariety.

= ideal of $\mathcal{N}$ is Poisson:

$$\{I_{\mathcal{N}}, \mathcal{O}_{\mathfrak{g}^*}\} \subseteq I_{\mathcal{N}}$$

$I_{\mathcal{N}} = (\mathcal{O}_{\mathfrak{g}^*}^{\mathfrak{g}})_+$ if $\mathfrak{g}$ s.s.

$$\mathcal{O}_{\mathfrak{g}^*}^{\mathfrak{g}} = \mathcal{O}_{\mathfrak{g}^*}^{G} \quad \mathrm{Lie}\, G = \mathfrak{g}^*$$

$\mathcal{N}$ is generically symplectic

$$\begin{array}{c} T^*(G/B) \\ \downarrow \\ \mathcal{N} \end{array}$$ is proper, birational, Poisson

preserves Poisson structure on sheaves of functions.

This implies $\mathcal{N} = \mathrm{aff}(T^*G/B) =: \mathrm{Spec}\,\Gamma(\mathcal{O}_{T^*G/B})$.

We now define $\begin{array}{c}\tilde{S}^0 \\ \downarrow \\ S^0\end{array}$. It also is a symplectic resolution

and S^0 is $\mathrm{aff}(\tilde{S}^0)$.

Cartoon: $e \in \mathcal{N}$ nilpotent element $\rightsquigarrow$ construct transverse slice S^0 to $\mathrm{Ad}G \cdot e \subseteq \mathcal{N}$

a) $e = 0$ $\quad S^0 = \mathcal{N} \quad \tilde{S}^0 = T^*G/B$

$\mathfrak{g} = sl_2$

$$\begin{array}{c} T^*\mathbb{P}^1 \\ \downarrow \\ \mathcal{N} \subseteq sl_2 \cong \mathbb{C}^3 \\ \| \\ \mathbb{C}^2/\{\pm I\} \end{array}$$

b) e = principal, i.e. $\mathrm{Ad}G \cdot e$ is open $\subseteq \mathcal{N}$
= regular

$$\Rightarrow \quad \tilde{S}^0 = pt$$
$$\downarrow$$
$$S^0 = pt$$

c) subregular e: $\mathrm{Ad}G \cdot e$ has codim 2

(NB. Whenever you have a symplectic resolution $\begin{array}{c}\downarrow \\ \mathcal{N}\end{array}$
you get a stratification of $\mathcal{N}$, orbits of Ham orbits (= Jordan forms for sl_n).)
$= \mathrm{Ad}\,G$ orbits

S^0 = "Kleinian" = "du Val" sing. if $\mathfrak{g}$ simple, simply-laced
= types ADE, $\quad \tilde{S}^0$ is a resolution.

There is a $\mathbb{C}^*$ dilating action on S°, $\tilde{S}^\circ$.

On S°, dilates to e, on $\tilde{S}^\circ$, dilates to $p^{-1}(e)$.

If $\mathfrak{g}$ is simple, simply laced (ADE)

$S^\circ \cong \mathbb{C}^2/_{\Gamma}$ $\quad \Gamma \subset SL_2\mathbb{C}$ finite

$\Gamma \xleftrightarrow[\text{McKay corresp.}]{}$ Dynkin diag. $(\mathfrak{g})$

$$\rho^{-1}(e) = \bigcup_{\substack{i \\ \text{vertices} \\ \text{of D.D.}}} \mathbb{P}^1_i, \quad \text{with } \mathbb{P}^1_i \cap \mathbb{P}^1_j = \begin{cases} * & i,j \text{ adjacent} \\ 0 & \text{else} \end{cases}$$

E.g. $\mathfrak{sl}_{n+1}$ case (A_n) $\quad \Gamma = \mathbb{Z}/_{n+1}$

$\bullet$—$\bullet$—$\bullet$ ... —$\bullet$ $\quad n$ vertices

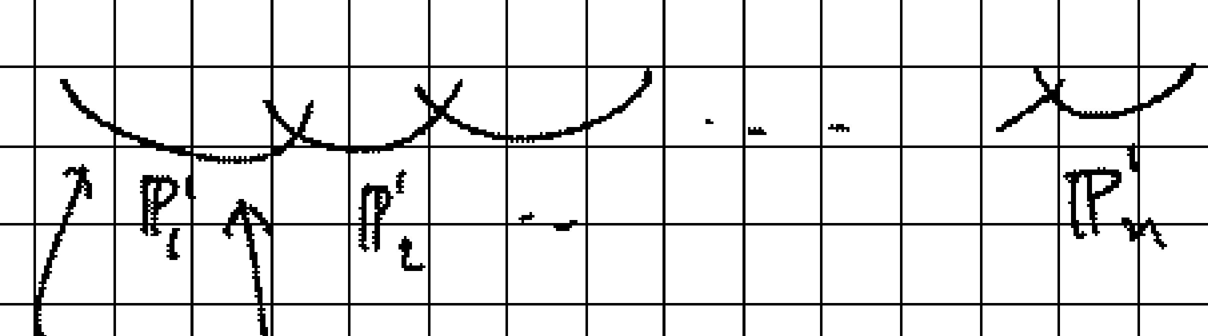
(nontrivial irreps of $\mathbb{Z}/_{n+1}$: there are n, via

$\tau_j : 1 \longmapsto \zeta^j$

$\zeta = e^{2\pi i/n+1}$

$\tau_i \otimes \mathbb{C}^2 \cong \tau_{i-1} \oplus \tau_{i+1}$.)

$\tilde{S}^\circ =$



$\mathbb{C}^*$ action

Not polarised but smooth symp dilates to $\rho^{-1}(e) = \bigcup_j \mathbb{P}^1_j$ (Weinstein).

$S^\circ = \mathbb{C}^2/_{\mathbb{Z}_{n+1}}$

$\mathbb{C}^*$ action dilates to 0.

Construction of $S^\circ = \mathcal{N} \cap (e + \ker \operatorname{Ad} f)$

$(e,h,f) \in \mathfrak{g}^3$, $\langle e,h,f \rangle \cong \mathfrak{sl}_2$

Jacobson-M $\Rightarrow$ such triple exists

S_e is transversal slice (affine lin space) to $\mathrm{Ad}\,G \cdot e$ at e.

Explicitly, $e = \left(\begin{array}{cccc|c} 0 & & & & \\ 1 & 0 & & & 0 \\ & & \ddots & & \\ & & 1 & 0 & \\ \hline & & 0 & & 0 \end{array}\right)$ $e_1, \dots, e_{n+1}$ std basis of $\mathbb{C}^{n+1}$.

2 Jordan blocks ($\Rightarrow$ subregular, codim = 2)

$p^{-1}(e) = \{(\mathcal{F}, e)\}$ $\mathcal{F}$ = complete flag

$$V_{n+1} \supsetneq V_n \dots \supsetneq V_1 \supsetneq 0$$
$$\begin{array}{ccccc} \Vert & & \cup & & \cup \\ U_{n+1} & \supset U_{n-1} & \cdots & U_0 & = 0 \end{array}$$

$\dim V_i = i$

s.t. $e(V_{i+1}) \subseteq V_i$ defn $U_{n-1} = e(U_{n+1})$

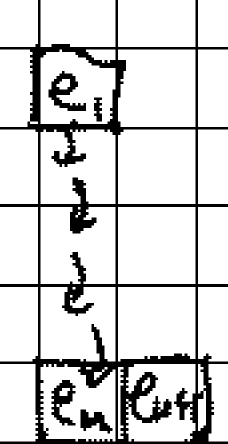
$e(U_{i+1}) = U_i \quad i < n.$

choosing such $V_i \iff$ picking j, setting

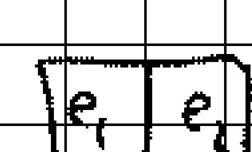
$e_{n+1} \in V_{j+1} \quad e_{n+1} \notin V_j$

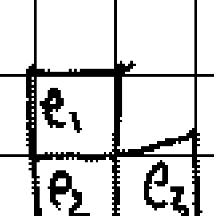
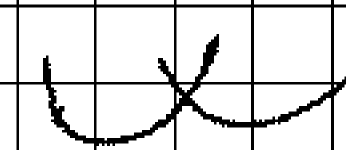
In this example

Picture: ~~kernel~~

kernel → 

choose flag so that this operation embeds flag in itself.

$n=1 \Rightarrow$  $\mathbb{P}^1$ 

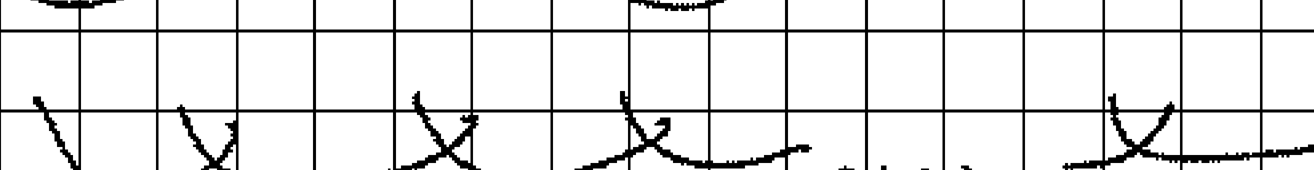
$n=2 \Rightarrow$  $\mathbb{P}^1 \vee_{pt} \mathbb{P}^1$ 

...

Choices of $V_j \cong \mathbb{P}^1 \setminus pt$

$\uparrow$ if $e_{n+1} \in V_j$

$\Rightarrow$ 

so  $= A_n$ singularity

Recap

ADE subreg. case

$$\begin{array}{c} \tilde{S}_e^\circ \\ \rho \big\downarrow \\ S_e^\circ \cong \mathbb{C}^2/\Gamma \end{array}$$

$\rho^{-1}(e) = \cup\cup \cdots \cup$

ρ isom. away from e.

Hamiltonian reduction

Idea: X = sympl $\qquad (T^*Y)$

$G \circlearrowright X$ Hamiltonian action $\Leftarrow (G \circlearrowright Y)$

produce symplectic quotient $X /\!/\!/ G$

Note: $T^*Y /\!/ G = T^*Y \to Y \to Y/G$ not. symp.

$\dim = 2\dim Y - \dim G$

Ham. red. replaces with $T^*(Y/G)$, if $G \circlearrowright Y$ free.

Moment map $\mathcal{H} : X \longrightarrow \mathfrak{g}^*$ $\quad \mathfrak{g} = \mathrm{Lie}\, G$

T^*Y: obtained from $\mathfrak{g} \to \mathcal{O}_{T^*Y}$
$\qquad\qquad\qquad\qquad\qquad \searrow \;\; \cup\!|$
$\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Vec}(Y)$

$\mathrm{Sym}\, \mathfrak{g} = \mathcal{O}_{\mathfrak{g}^*}$

Take a char. $\chi \in \mathfrak{g}^*$ (G-invt.)

$$X /\!/\!/_\chi G := \mathcal{H}^{-1}(G \cdot \chi) /\!/ G$$

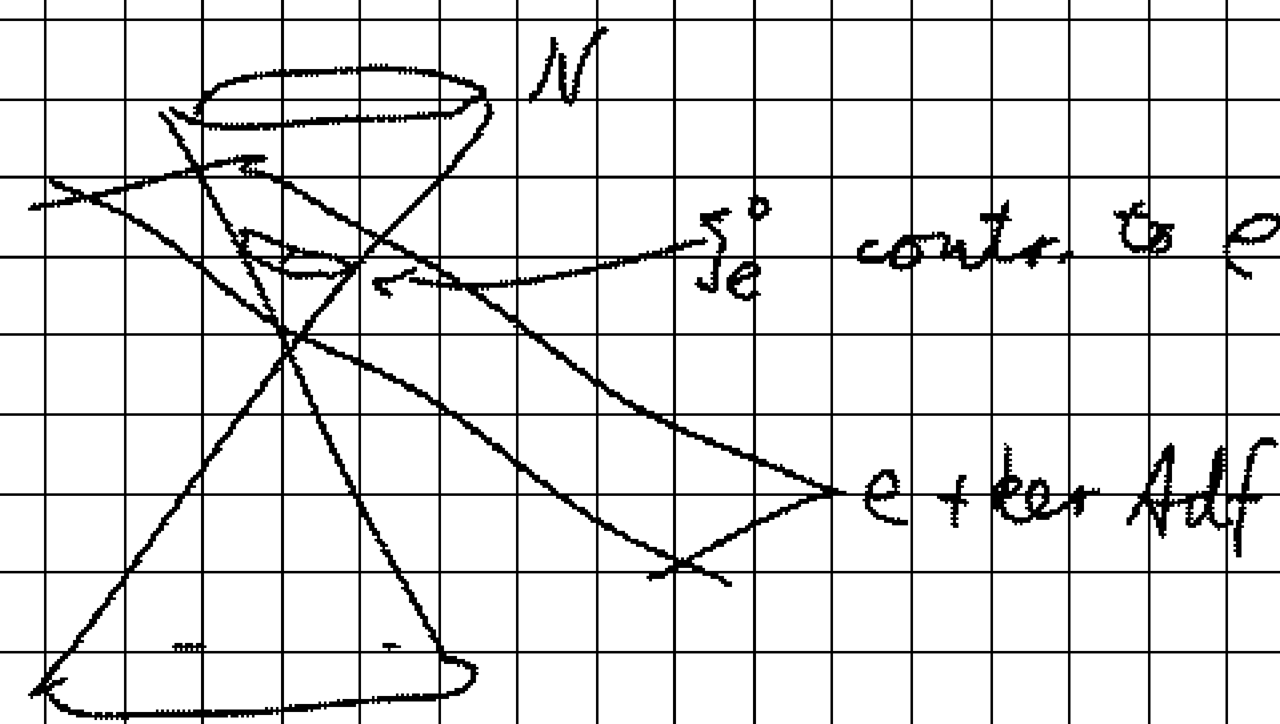
often $\chi = 0$

$$\mu^{-1}(0)/\!/G$$

To get $e + \ker \mathrm{Ad} f$: we ~~look~~ up $\overset{\text{nilp}}{\mathfrak{m}} \leq \mathfrak{g}$

$$S_e^0 = \mu^{-1}(\chi)/\!/M \qquad \text{Lie } M \subset \mathfrak{m}.$$

$S_e^0 =$



$$\begin{array}{ccc} e \in & \mathcal{N} & \\ \downarrow & \cap & \\ \chi & \mathfrak{g} & \\ & \wr\!\!\parallel & \\ & \mathfrak{g}^* & \end{array}$$

$M \leq G$ acts freely on e.

$\mathrm{stab}_{M_\chi}(e) = \{1\}$

$\mathfrak{m}_\chi$ s.t.

$$\mathfrak{m} \oplus \ker(\mathrm{ad} f) \cong \mathfrak{m}_\chi^\perp = \mu^{-1}(0).$$