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Note Title 6/16/2011

G complex Lie group semi-simple ($G = SL_2\mathbb{C}$)

$\mathfrak{g}$ Lie algebra of G

Adjoint action $Ad: G \circlearrowright \mathfrak{g}$

$$\mathfrak{g} = \underset{\text{pos. roots}}{\mathfrak{n}_+} \oplus \underset{\text{Cartan}}{\mathfrak{h}} \oplus \underset{\text{neg. roots}}{\mathfrak{n}_-}$$

E.g. $\mathfrak{g} = \mathfrak{sl}_2$ = traceless 2×2 matrices

$\mathfrak{n}_+$ = upper triangular

$\mathfrak{h}$ = diagonal

$\mathfrak{n}_-$ = lower triangular

$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

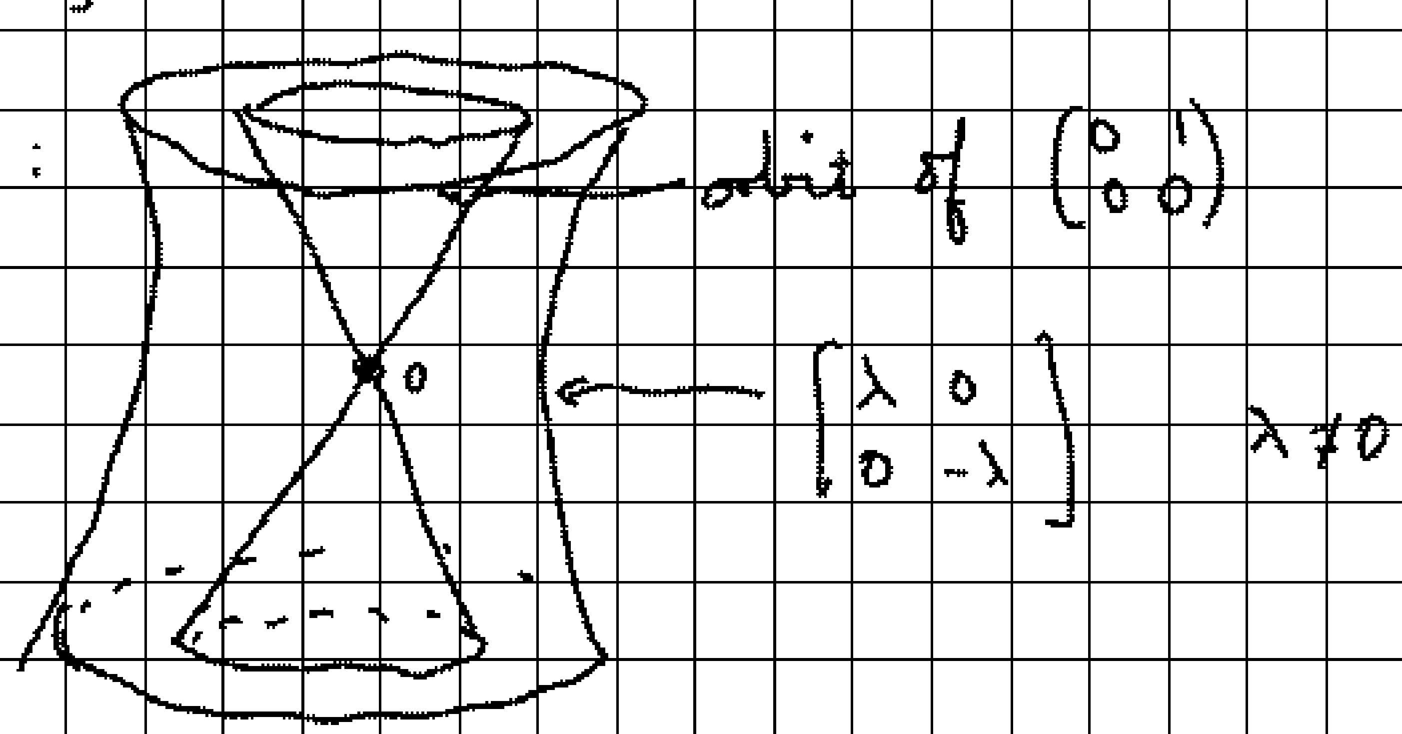
$\mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \simeq \mathfrak{h}$

$\mathfrak{h}$ canonical, $\mathfrak{n}_+, \mathfrak{n}_-$ aren't

Defn: $X \in \mathfrak{g}$ is <u>regular</u> if $\dim \underbrace{Z_{\mathfrak{g}}(X)}_{\text{centralizer}} = \dim \mathfrak{h}$

(NB for all X, $\dim \geq \dim \mathfrak{h}$)

Ex: $(\mathfrak{sl}_2)_{reg} = \mathfrak{sl}_2 - 0$.

Picture: 

$X \in \mathfrak{g}$ is semisimple if $ad\,X$ is diagonalizable

$\mathfrak{sl}_n$: diagonalisable matrices.

Weyl group: $H \subset G$ Cartan

$$\Rightarrow W_G = N_G(H)/H$$

$W_{sl_n} = S_n$ = action by permuting eigenvalues.

$\exists$ natural map

$$\mathfrak{g} \longrightarrow \mathfrak{h}/\!\!/W$$

take $\mathfrak{h} = \operatorname{Spec} R$, $\mathfrak{h}/\!\!/W = \operatorname{Spec} R^W$ (R^W = W-invariant functions)

$X \longmapsto$ unordered set of eigenvalues.

<u>Thm</u> (Chevalley)

$$\mathfrak{g}/\!\!/G \cong \mathfrak{h}/\!\!/W$$

Flag variety G/B

E.g. sl_2: $G/B = \mathbb{P}^1$

$G/B = \mathcal{B}$ space of Borels: B is just exponentiating $\mathfrak{b}$

$g \longmapsto g^{-1} B g$

sl_n Borel: matrices upper triangular w.r.t. a flag.

$$0 = V_0 \subsetneq V_1 \dots \subsetneq V_n = V$$

(can take this as definition for sl_n).

<u>Thm</u>: (Beilinson – Bernstein)

$D\text{-mod}(G/B) \cong (\mathfrak{g}\text{-reps})_0$ centraliser $= \mathcal{U}_0\text{-mod}$

$(\mathfrak{g} \to \operatorname{Vect}(G/B))$

$\Rightarrow \underset{\text{env. alg.}}{\mathcal{U}_{\mathfrak{g}}} \twoheadrightarrow D_{G/B}$ with kernel $\mathfrak{Z}_{\mathfrak{g}}$

$$\mathcal{U}_0 = \mathcal{U}_{\mathfrak{g}}/\mathfrak{Z}_{\mathfrak{g}^+}$$

$$D\text{-mod}(G/B)_{rh} \cong D^b_{c.c}(G/B) \cong \mathrm{Fuk}_{c.c}(T^* G/B)$$

NB. Any irred. rep. corresponds to a regular holonomic D-module.

$\tilde{\mathfrak{g}}$ Grothendieck resolution

$$\tilde{\mathfrak{g}} = \{ b \in G/B,\ X \in b \}$$

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl] \arrow[dr] & \\ \mathfrak{g} & & G/B \end{tikzcd}$$

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\tilde{q}"] \arrow[d, "\mu_{\mathfrak{g}}"'] & \mathfrak{h} \arrow[d, "s"] \\ \mathfrak{g} \arrow[r, "q"'] & \mathfrak{h}/\!\!/W \end{tikzcd}$$

(Analogy: matrix has eigenvalues, but not in a given order - choose a flag to get an ordering of eigenvalues.)

is commutative, but not a fibre square.

But, restricting to regular, semi-simple elements

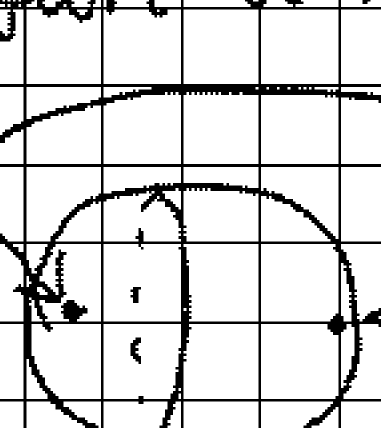
$$\begin{tikzcd} \tilde{\mathfrak{g}}_{reg,ss} \arrow[r] \arrow[d] & \mathring{\mathfrak{h}} \arrow[d] \\ \mathfrak{g}_{reg,ss} \arrow[r] & \mathring{\mathfrak{h}}/\!\!/W \end{tikzcd}$$

W - covering space $\Rightarrow$ so is $\begin{array}{c} \tilde{\mathfrak{g}}_{reg,ss} \\ \downarrow \\ \mathfrak{g}_{reg,ss}. \end{array}$

matrices with distinct eigenvalues

E.g. $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} \in \mathfrak{g}_{reg,ss}$

fibre over this point $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$

$\mathbb{P}^1$ 

X is a vector field on G/B

$(X, b) \rightsquigarrow$ vector field with choice of critical point of X.

Nilpotent cone: $\mathcal{N} = q^{-1}(0)$ = nilpotent elts in $\mathfrak{g}$

E.g. for $\mathfrak{sl}_2$, $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ s.t. $a^2 + bc = 0$.

$\tilde{\mathcal{N}} = \tilde{q}^{-1}(0)$ is called the Springer resolution of $\mathcal{N}$

$\tilde{\mathcal{N}}$ is a symplectic resolution of $\mathcal{N}$ (i.e. it's a resolution as varieties, and $\tilde{\mathcal{N}}$ is symplectic).

$\tilde{\mathcal{N}} = \{(x,b) \in \mathcal{N} \times G/B : x \in b\}$

<u>Fact:</u> $\tilde{\mathcal{N}} = T^*(G/B)$ so it's manifestly symplectic.

Why?

$T^*(G/B) = G \times_B b^\perp$

$T_B(G/B) = \mathfrak{g}/b$

$$\begin{array}{ccc} T^*_B(G/B) = & b^\perp & \subset \mathfrak{g}^* \\ & \wr\! \| & \wr\! \| \longleftarrow \text{via Killing form} \\ & n \subset b & \subset \mathfrak{g} \end{array}$$

$$\begin{tikzcd} G/B & T^*(G/B) \cong \tilde{\mathcal{N}} \arrow[l, "p"'] \arrow[r, "\tilde{i}"] \arrow[d, "\mu_{\mathcal{N}}"'] & \tilde{\mathfrak{g}} \arrow[r, "\tilde{q}"] \arrow[d, "\mu_{\mathfrak{g}}"'] & \mathfrak{h} \arrow[d, "s"] \\ & \mathcal{N} \arrow[r, "i"'] & \mathfrak{g} \arrow[r] & \mathfrak{h}/W \end{tikzcd}$$

The fundamental diagram of representation theory

<u>Springer theory</u>

(How to produce all representations of W (e.g. S_n) geometrically)

or $*$ (proper)

Define $S_{\mathfrak{g}} = \mu_{\mathfrak{g}!}\, \mathbb{C}_{\tilde{\mathfrak{g}}}\, [\dim \mathfrak{g}]$

$S_{\mathcal{N}} = \mu_{\mathcal{N}!}\, \mathbb{C}_{\tilde{\mathcal{N}}}\, [\dim \mathcal{N}]$

$\mu_{\mathfrak{g}!}$ says: take vector field X on G/B, put the cohomology of its zero set as a stalk over X.

Clearly, $S_{\mathcal{N}} \cong i^* S_{\mathfrak{g}} [\dim \mathfrak{h}]$.

Two facts:

1) There is an action of W on $S_{\mathfrak{g}}$, $S_{\mathcal{N}}$

NB. This is clear on $\mathfrak{g}_{reg,ss}$ etc... just a W-covering space. But incredibly this action extends over all these blown-up fibres etc.

$$\mathcal{E}nd(S_{\mathfrak{g}}) = \mathbb{C}[W] \qquad (\mathcal{E}nd = R^0 \mathcal{E}nd)$$

2) $S_{\mathfrak{g}}$ and $S_{\mathcal{N}}$ are Fourier transforms of each other

$$\mathfrak{g} \simeq \mathfrak{g}^*$$
$$S_{\mathfrak{g}} \quad S_{\mathcal{N}}$$

Fourier Transform : $D_{c,\text{conical}}(V) \xrightarrow{\sim} D^b_{c,\text{conical}}(V^\vee)$

for any vector space V, with conical being w.r.t. $\mathbb{R}_+$ action.

To see (1) from Fukaya category perspective,

$$\begin{array}{ccccccccc} T^*\mathfrak{g} & \cong & \mathfrak{g} \times \mathfrak{g}^* & \cong & \mathfrak{g} \times \mathfrak{g} & \xrightarrow{p_2} & \mathfrak{g} & \longrightarrow & \mathfrak{h}/W \\ \cup & & & & & & & & \Psi \\ L_\lambda \subset P_\lambda & & & \longrightarrow & & & & & \lambda \text{ regular} \end{array}$$

P_λ = inverse image of λ.

We define $L_\lambda = P_\lambda \cap \{ \mathcal{G}^{-1}(0) \}$

where $\mathcal{G}(g, \xi) := [g, \xi]$ under $\mathfrak{g}^* \to \mathfrak{g}$

Claim: $L_\lambda \subset T^*\mathfrak{g}$ is Lag. brane

Thm:
$$\begin{array}{ccc} L_\lambda & \in & \mathcal{F}(T^*\mathfrak{g}) \\ \updownarrow & & \updownarrow \\ S_{\mathfrak{g}} & \in & D_c(\mathfrak{g}) \end{array}$$

Now moving λ around in $\pi_1(\mathring{\mathfrak{h}}/\!/W) \to W$ gives a W-action on this Lagrangian brane L_λ.

To see (2): Fourier transform is just switching

$$X \times X \longrightarrow X \times X$$
$$x_1, x_2 \longmapsto x_2, x_1.$$

in symplectic language

~~M~~ S_N is conormal to an orbit.