James Pascaleff - Mirror Symmetry for Toric Varieties

Note Title 6/16/2011

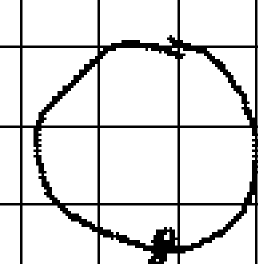
Homological Mirror Symmetry

$$D^b \mathrm{Coh}(X) \simeq D^b \mathrm{Fuk}(?)$$

Case of $\mathbb{P}^1$

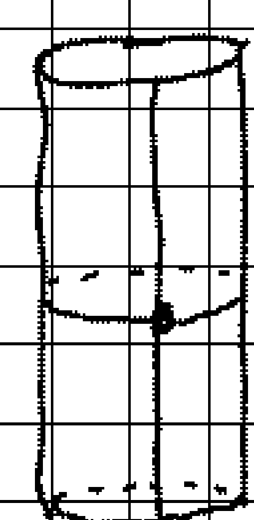
$D^b\mathrm{Coh}(\mathbb{P}^1) = \mathrm{Perf}(\bullet \rightrightarrows \bullet)$ (Beilinson)

Note that $\bullet \rightrightarrows \bullet$ $\simeq$ exit path category of the stratification of S^1:



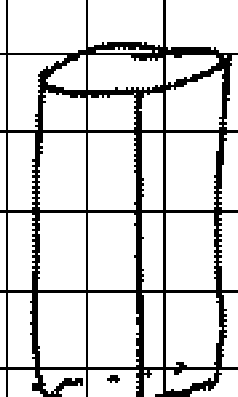
$D^b(\mathrm{coh}\,\mathbb{P}^1) \simeq D^b_{c,s}(S^1) \simeq \mathrm{Fuk}(T^* S^1, \Lambda_s)$

not canonical
(we'll make extra choices to fix this)

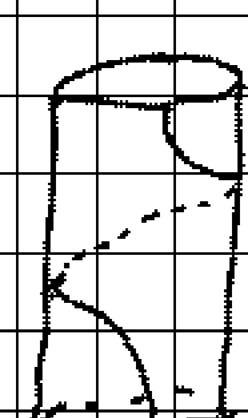


Some branes:

$\mathcal{O}_{\mathbb{P}^1} \longleftrightarrow i_* \mathbb{C}_x \longleftrightarrow$

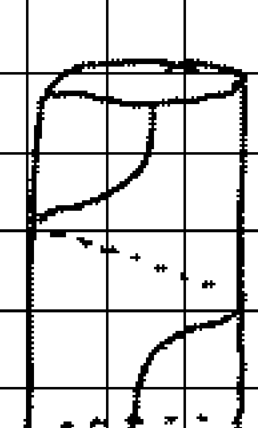


$\mathcal{O}_{\mathbb{P}^1}(-1) \longleftrightarrow j_* \mathbb{C}_{S^1 \setminus x} \longleftrightarrow$



standard object

$\mathcal{O}_{\mathbb{P}^1}(1) \longleftrightarrow j_! \mathbb{C}_{S^1 \setminus x} \longleftrightarrow$

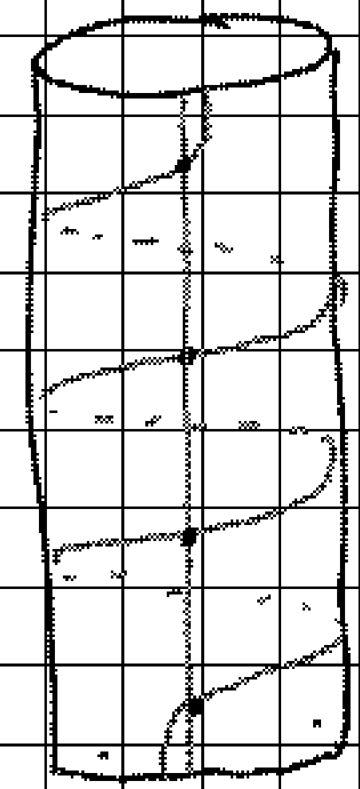


costandard object

$\mathcal{O}_{\mathbb{P}^1}(n) \longleftrightarrow$ ... (ex) $\longleftrightarrow$

$\mathcal{L}(n)$

$\mathrm{Hom}(\underline{\mathcal{O}}, \underline{\mathcal{O}(n)}) =$

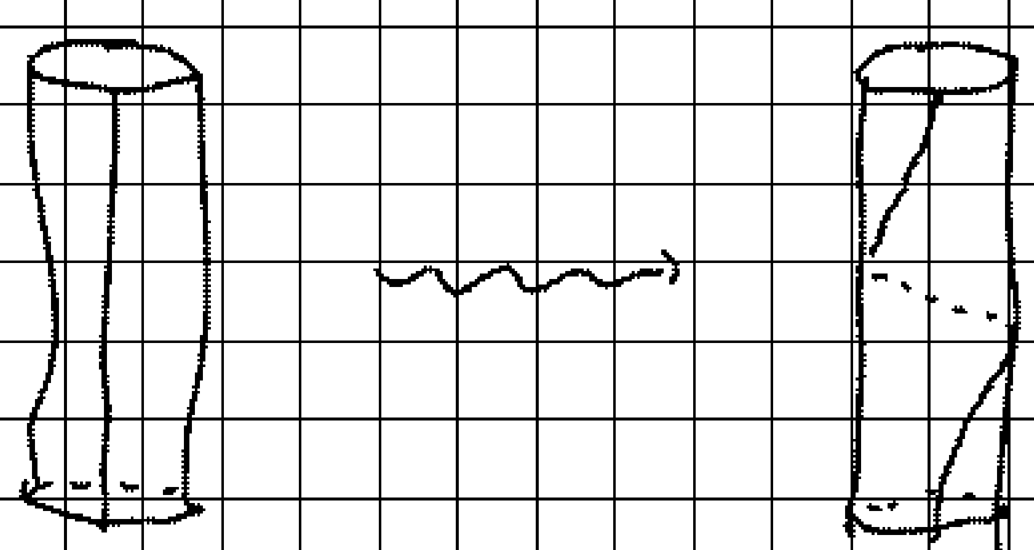


$n+1$ intersections

$\cong$ degree n homogeneous polynomials in 2 variables

We can check that this identification is compatible with multiplication in the Fukaya category.

Dehn twist $\varphi: T^*S^1 \to T^*S^1$ identity at ∞ sends

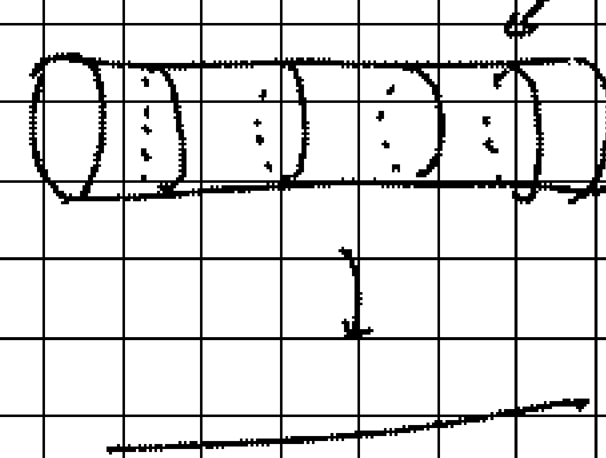


this is mirror to $\otimes\mathcal{O}(1)$

ex: What is the integral transform (sheaf on $S^1 \times S^1$) that corresponds to the Dehn twist?

We have two projections:

$$\begin{array}{ccc} T^*S^1 & \longrightarrow & S^1 \\ \downarrow & & \\ \mathbb{R} & & \end{array}$$

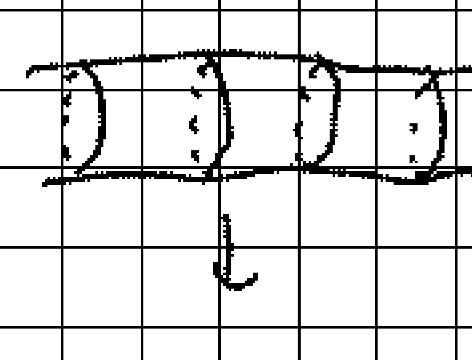


some fibres are not exact - but equivalent to 0-section with non-unitary local system.

all of the Lagrangians $\mathcal{L}(n)$ are sections of this one.

T-duality:

$T = \mathbb{C}^* \subset \mathbb{P}^1$ $\qquad$ $\mathbb{P}^1 \setminus \{0, \infty\} =$



$T_{\mathbb{R}}$ orbits

$T_{\mathbb{R}} = S^1$

$\mathbb{R}$

There are dual torus fibrations: $N = \mathbb{Z}$, $N_{\mathbb{R}} = \mathbb{R}$, $M = \mathrm{Hom}(N, \mathbb{Z})$

$$\begin{array}{ccc} \mathbb{P}^1 \setminus \{0,\infty\} & & T^*S^1 \\ \downarrow & \text{fibres} = T_{\mathbb{R}} = N_{\mathbb{R}}/N & \downarrow \quad \text{fibres} = M_{\mathbb{R}}/M \\ N_{\mathbb{R}} & & N_{\mathbb{R}} \qquad = T_{\mathbb{R}}^{\vee} \end{array}$$

$T_{\mathbb{R}}^{\vee} \cong$ flat $U(1)$-bundle on $T_{\mathbb{R}}$.

$\mathcal{O}_{\mathbb{P}^1}(n)$ choose a Hermitian metric h which is $T_{\mathbb{R}}$-invariant

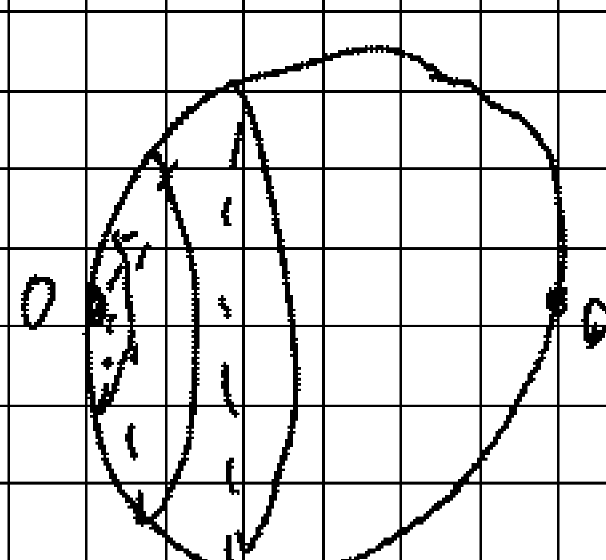
Let ∇_h be Hermitian connection compatible w/ holomorphic structure.

Get $(\mathcal{O}_{\mathbb{P}^1}(n), \nabla_h)\big|_{T_{\mathbb{R}}\text{-orbit}}$ is a flat $U(1)$ connection

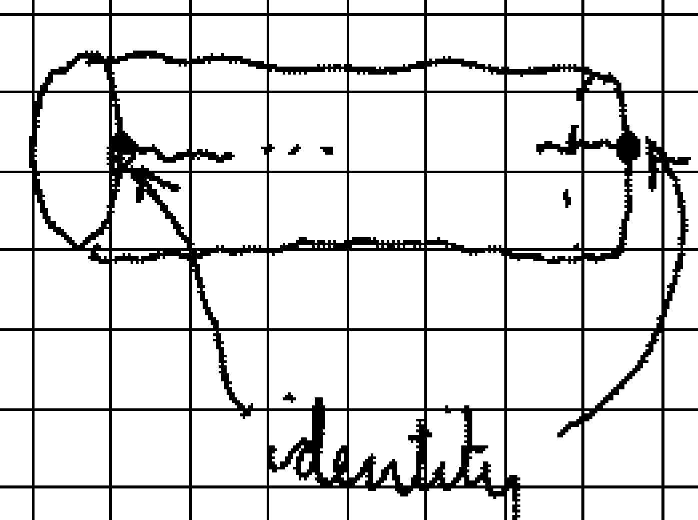
$\Rightarrow$ element of $T_{\mathbb{R}}^{\vee}$

$\Rightarrow$ this defines the Lag. section mirror to $\mathcal{O}_{\mathbb{P}^1}(n)$.

For the line bundle with connection $(\mathcal{O}_{\mathbb{P}^1}(n), \nabla_h)$ to be extendable to $\{0, \infty\}$ the monodromy must converge to the identity at $0, \infty$:



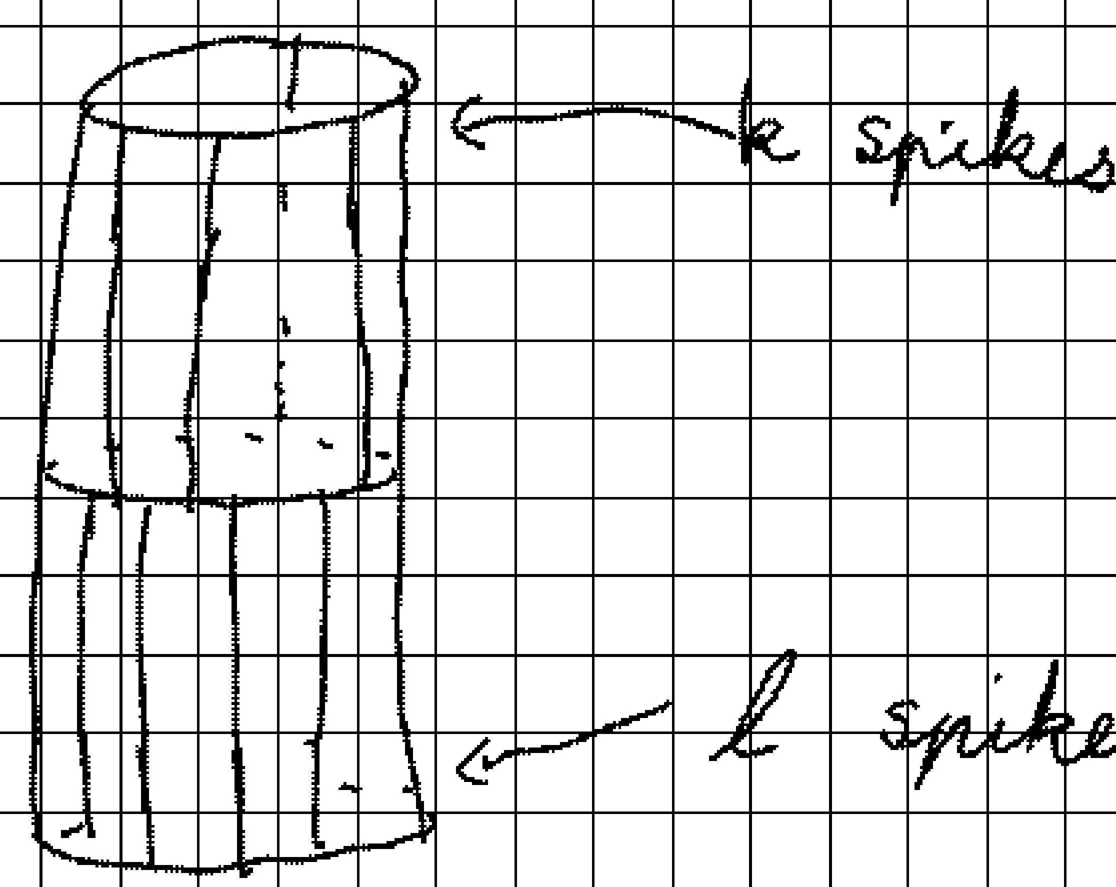
$\Rightarrow$ the Lagrangian L converges to a fixed direction at $\pm\infty$:



Moral: identifying $T_{\mathbb{R}}^{\vee} = \mathrm{Hom}(\pi_1 T_{\mathbb{R}}, U(1))$ with the fibres gives a section (identity).

Another example:

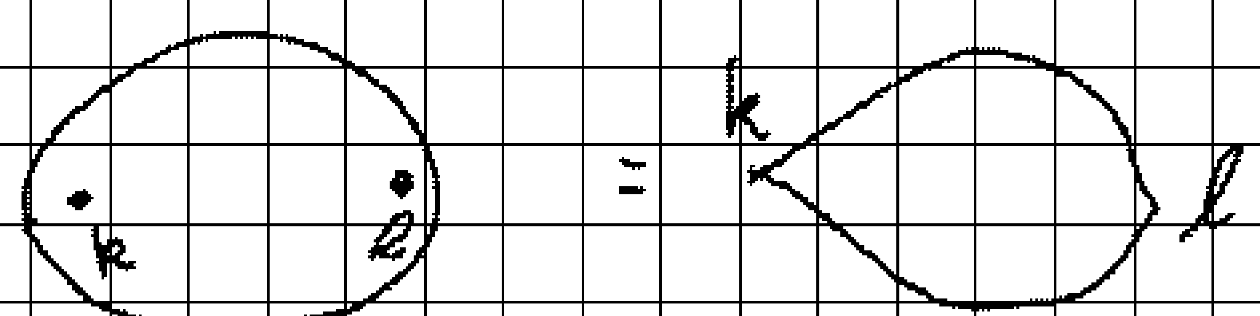
$\Lambda_{k,\ell} =$



$D^b_{\Lambda_{k,\ell}}(S^1) \subset D^b_c(S^1)$

not nec. constructible w.r.t. any stratification, just have sing. support in $\Lambda_{k,\ell}$.

Having k spikes means monodromy of your connection is a kth root of unity $\Rightarrow$ your manifold has a k-fold orbifold point:



$D^b(\mathrm{coh}(\mathbb{P}^1_{k,\ell})) \cong \mathrm{Fuk}_{\Lambda_{k,\ell}}(T^* S^1)$.