$$\begin{array}{ccc} \text{standard objects} & \xrightarrow[\mathbb{C}_Y \longmapsto L_Y]{} & Fuk(T^*X) \\ \cap & & \updownarrow \\ D^b_c(X) & \overset{\mu_X}{\dashrightarrow} & D^b Fuk(T^*X) = \mathcal{F}(T^*X) \\ \| & & \\ \text{triangulated envelope} & & \end{array}$$

μ_X is a quasi-embedding

Goal: μ_X is a quasi-equivalence

$\mathcal{F}(T^*X) := \{$twisted complexes in $Mod(Fuk(T^*X))$ built out of $Hom_{Fuk}(-, L)$ $\forall L$ branes$\}$

Goal: express any $Hom_{Fuk}(-, L)$ as a tw. cpx of $Hom(-, L_Y)$

$\forall L$, $\forall P \in Fuk$ (test object) want to write $Hom(P, L)$ in terms of $Hom(P, L_Y)$. (step 1).

Motivation: X scheme, $P, L \in Coh(X)$

$$\begin{aligned} Hom_{Coh(X)}(P, L) &\cong Hom(P \otimes L^\vee, \mathcal{O}_X) \\ &\cong Hom(\Delta^*(P \boxtimes L^\vee), \mathcal{O}_X) \\ &\cong Hom(P \boxtimes L^\vee, \underbrace{\Delta_* \mathcal{O}_X}_{\mathcal{O}_\Delta}) \end{aligned}$$

$\Delta : X \to X \times X$ diagonal

$\boxtimes \rightsquigarrow \times$ product

$\vee \rightsquigarrow \alpha_X$ multiply by -1 on T^*X.

$$(L, \mathcal{E}, \tilde{\alpha}, b) \xmapsto{\alpha_X} (a(L), a^*(\mathcal{E}^\vee), -d_X - a^*(\tilde{\alpha}), a^*(b)).$$

$$\mathcal{O}_\Delta \rightsquigarrow \alpha_{X\times X}(L_{\Delta_X}) = \alpha_{X\times X}\left(\mu_{X\times X}(i_* \mathbb{C}_{\Delta_X})\right)$$

__Lemma:__ $\mathrm{Hom}_{\mathcal{F}(T^*X)}(P, L) \cong \mathrm{Hom}_{\mathcal{F}(T^*(X\times X))}(P \times \alpha(L), \alpha(L_{\Delta_X}))$

(all the moduli spaces are the same, on the nose).

__Lemma:__ $P, P' \in \mathcal{F}(T^*X_0)$, $Q, Q' \in \mathcal{F}(T^*X_1)$

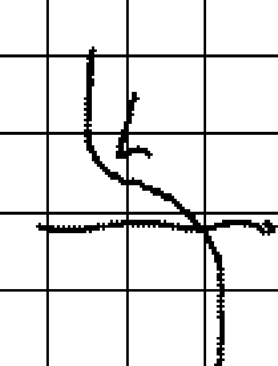
$$\mathrm{Hom}_{\mathcal{F}(T^*(X_0\times X_1))}(P\times Q, P'\times Q') \cong \mathrm{Hom}_{\mathcal{F}(T^*X)}(P, P') \otimes \mathrm{Hom}_{\mathcal{F}(T^*X)}(Q, Q')$$

We want to approximate $\Delta_X \subset X \times X$ by a "staircase":

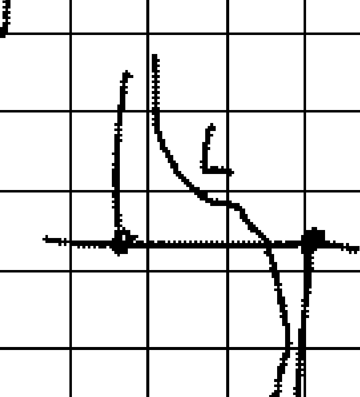
triangulate X $\quad T = \{t_\alpha \mid \alpha \in A\}$

Condition: $L^\infty \subset \overline{\bigsqcup_\alpha T^*_{t_\alpha} X}^\infty$ (given L)

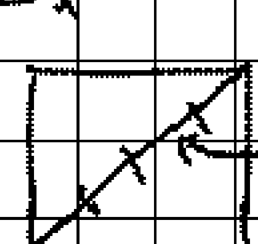
E.g. if



we choose T:



Since $i_* \mathbb{C}_{\Delta_X}$ is a twisted complex built out of

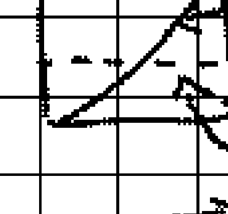
$i_* \mathbb{C}_{\Delta_\alpha}$:  Δ_α = diagonal in $t_\alpha \times t_\alpha$

$\alpha(L_{\Delta_X})$ is a twisted complex of the $\mu_{X\times X}(i_* \mathbb{C}_{\Delta_\alpha})$

$\Rightarrow \mathrm{Hom}_{\mathcal{F}(T^*(X\times X))}(P \times \alpha(L), \alpha(L_{\Delta_X}))$

can be expressed as a tw cpx of

$\mathrm{Hom}(P \times \alpha(L), L_{\Delta_\alpha})$

Now we would like to replace  L_{Δ_α} replace by this

$\Rightarrow$ staircase

i.e. $L_{\Delta_\alpha} \cong L_{\tau_\alpha} \times L_{\{c_\alpha\}}$

conormal to point in centre of τ_α

If this were true (it's not), we would get

$$\mathrm{Hom}(P \times \alpha(L), L_{\Delta_\alpha}) \cong \mathrm{Hom}(P \times \alpha(L), L_{\tau_a} \otimes L_{\{c_a\}})$$

$$\cong \mathrm{Hom}(P, L_{\tau_a}) \otimes \mathrm{Hom}(\alpha(L), L_{\{c_a\}})$$

$\Rightarrow$ can build $\mathrm{Hom}(P, L)$ out of $\mathrm{Hom}(P, L_{\tau_a})$, tensored with some vector spaces which are independent of P.

$\Rightarrow$ By Yoneda Lemma, original L is a tw. cpx. of L_{τ_a}.

<u>But</u> it is not true.

Instead look at homotopy $\Delta_{a,s}$ between Δ_a and $t_a \times \{c_a\}$

Define $L_{\Delta_{a,s}} = \mu_{X \times X}(i_* \mathbb{C}_{\Delta_{a,s}})$

<u>Non-char def. lemma</u>: If $Q \in \mathcal{F}(T^*X)$, family of branes $\mathcal{L}_s$, $s \in \mathbb{R}$,

if $\overline{Q}^\infty \cap \overline{\mathcal{L}_s}^\infty = \phi$,

Then $\mathrm{Hom}(Q, \mathcal{L}_s)$ is indep. of s.

So we need to choose the triangulation so that this is satisfied, for $Q = P \times \alpha(L)$, $\mathcal{L}_s = L_{\Delta_{a,s}}$

It turns out you can do this and it doesn't depend on P (you only move $\Delta_{a,s}$ in the direction orthogonal to P direction).