Anatoly Preygel - Infinitesimal Fukaya Category

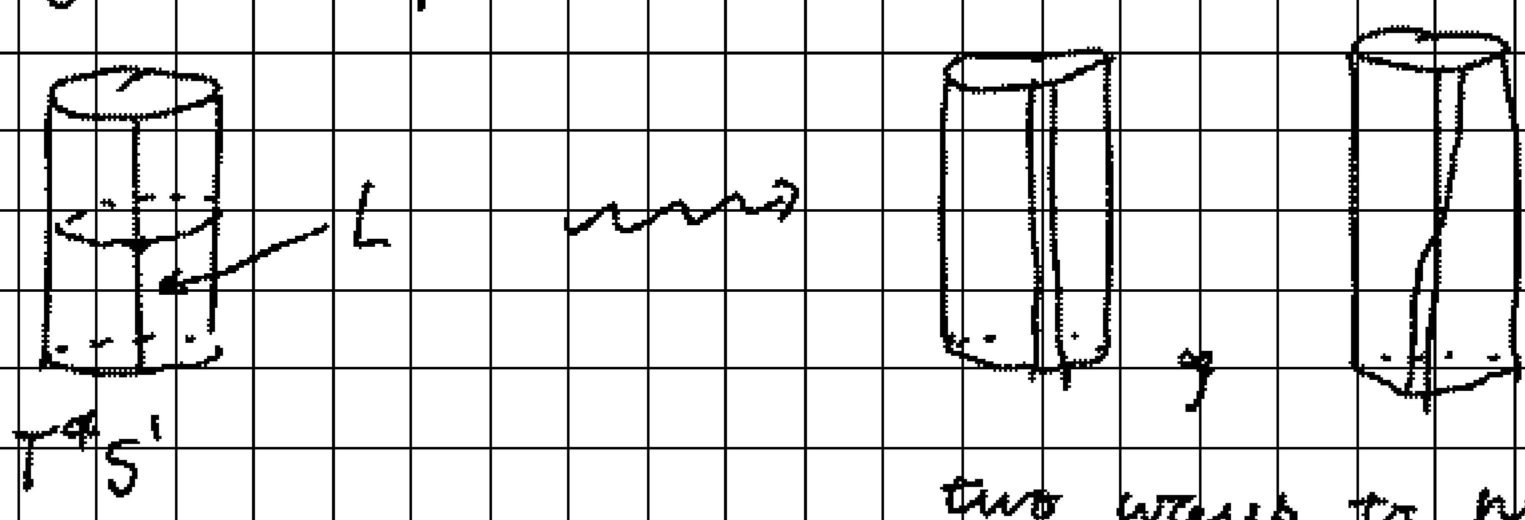
Note Title 6/15/2011

§0. Reminder: (N, ω) exact, Liouville (e.g. T^*M)

$\rightarrow$ Fuk (N), now containing non-compact Lagrangians.

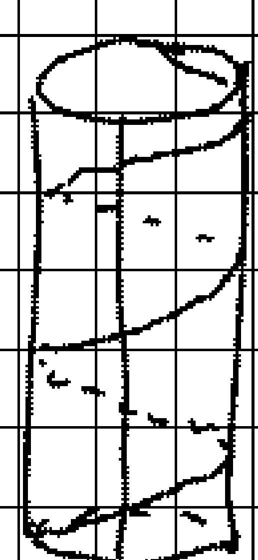
How do we perturb Lagrangians that ~~are~~ intersect at ∞

① arbitrary small perturbation? <u>No</u>:



two ways to push L off itself give two different answers

② wrapped: 'wrap' L by a Hamiltonian which is quadratic at ∞ (geodesic flow)



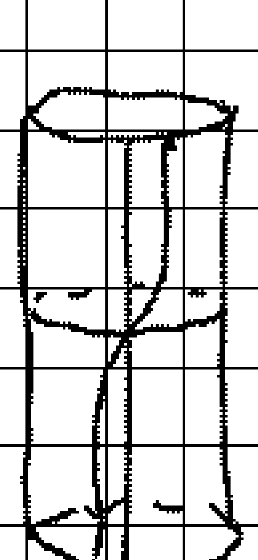
$$\mathrm{Hom}(L,L) = k[\mathbb{Z}] = C_* \Omega_{pt} S^1$$

<u>Thm</u>: $\mathrm{Fuk}^{wr}(T^*M, \pi_M^* w_2 TM) \simeq \mathrm{Sh}_{\substack{loc\\const}}(M, \mathrm{Cpx}_k)$

$$\supset D_{wc}(M)$$

constructible sheaves of complexes$_*$ (i.e. don't require finite rank condition)

③ infinitesimal: wrap L by "normalised geodesic flow".



$$\mathrm{Hom}(L,L) = k = \mathrm{Hom}_{Sh_c}(i_! \underline{k}, i_! \underline{k})$$

$$i : pt \hookrightarrow S^1.$$

Key issues

1) Given ≥ 2 Lagrangians, need to ensure (after perturbation)

- $\cap$ points stay away from ∞
- hol disks stay away from ∞.

2) Morse $\longleftrightarrow$ $\mathcal{F}uk^{inf}$.

§1.

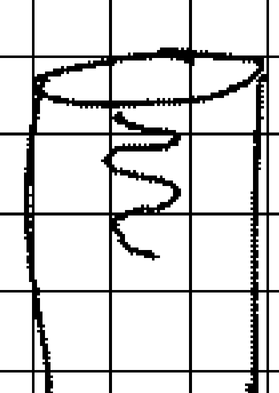
$$\begin{array}{ccccc} T^*M & \subset & \overline{T^*M} & \supset & (T^*M)^\infty \\ \wr\wr & & \wr\wr & & \wr\wr \\ \mathring{D}^*M & \subset & \bar{D}^*M & \supset & S^*M \end{array}$$

open disk bundle

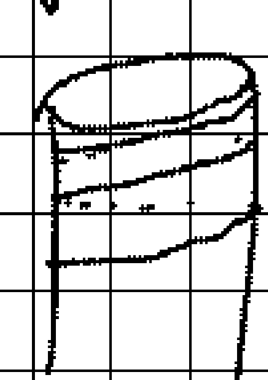
Defn: A brane which is an object of $\mathcal{F}uk^{inf}$ will be an exact Lagrangian $L \subset T^*M$ (+ brane structure) s.t.

- $\bar{L} \subset \overline{T^*M}$ is a subanalytic subset

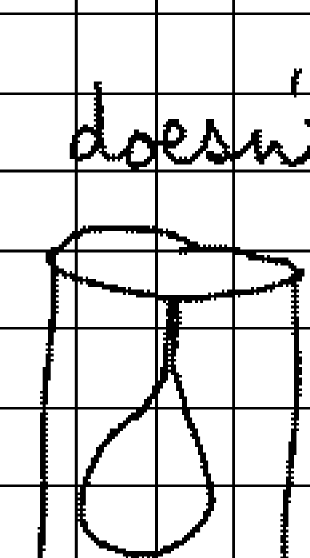
(this rules out things like



or



but doesn't mean it's conical near ∞, e.g.



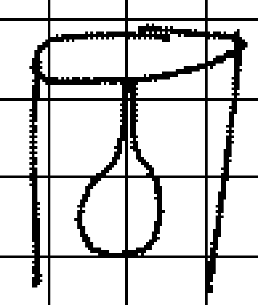
is OK). (this bounds intersection points)

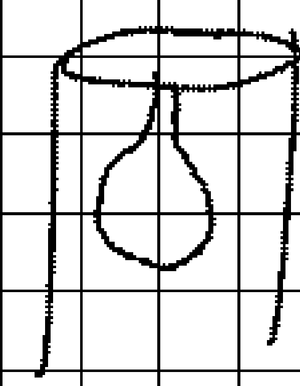
- L has a suitable perturbation which is symplectically tame.

Defn: L is symplectically tame if:

- $\exists\ r_L > 0$ s.t. $B_r(x) \cap L$ contractible $\forall x \in L$, $r < r_L$.

• $\exists\, C_L \quad d_L(x,y) \leq C_L\, d(x,y)$

E.g.  fails both of these.

But there's a nice perturbation 

that makes it symp. tame.

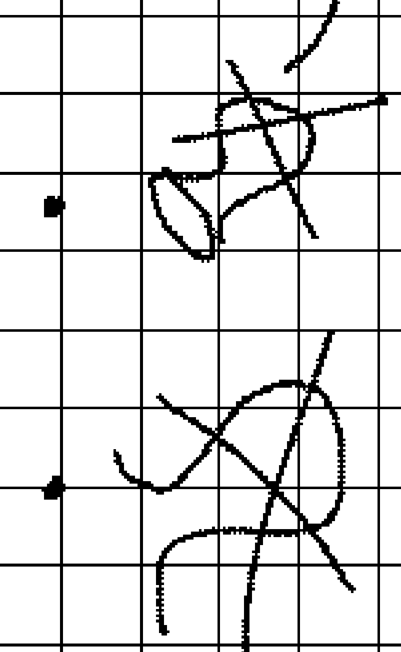
E.g. conormal bundle of $\{y = x^3\} \subset \mathbb{R}^2$ w.r.t. Sasakian metric, fails the second condition.

NB: $(T^*M)^\infty$ contact
$\cup$
$(L)^\infty$ isotropic

Reeb flow = norm. geod. flow <u>will separate</u> $(L_0)^\infty$, $(L_1)^\infty$

<u>NB:</u> Exactness $\Rightarrow$ A priori area bound $\Rightarrow$ Diameter bound $\Rightarrow$ Compactness of moduli space of hol. curves

Stokes



(Sikorav)

To define μ^n, roughly, we perturb Lagrangians $L_0, \dots, L_n$ by times $t_0 < t_1 < \dots < t_n$
(first choose t_n small, then t_{n-1} much much smaller, then t_{n-2}, etc.)

Way of getting objects:

<u>(Open) std branes</u>

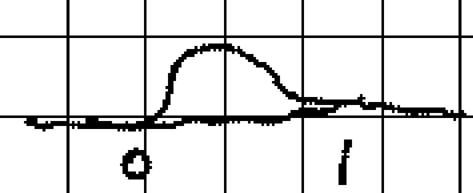
$\mathcal{U} \subset M$, m a defining function (cf. Thomas K's talk)

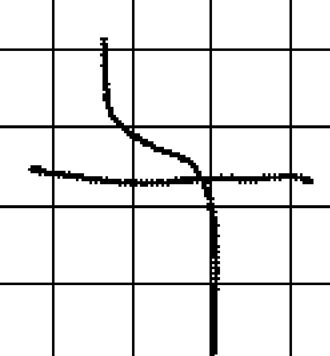
$\Rightarrow \Gamma(d\log(m))$, is a non-compact Lagrangian.

$\shortparallel$

$L_{\mathcal{U},f}$ where $f = \log(m)$.

E.g. $\mathcal{U} = (0,1)$

$\Rightarrow m =$ 

$\Rightarrow d\log m =$ 

If $Y \hookrightarrow M$

m defining function for ∂Y

$\Rightarrow L_{Y,f} = T^*_Y M + \Gamma_{d\log m}|_{m>0}$ hybrid Lagrangian (see Ali's talk)

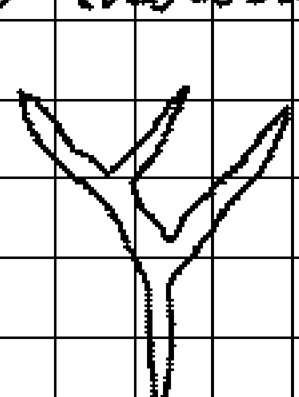
So we've defined, for each object of $\mathrm{Mor}(M)$, an object of $\mathfrak{Fuk}^{\inf}(T^*M)$.

Furthermore, the Hom spaces are naturally in bijection:

$\mathrm{Crit}(f_1 - f_0) \cong \Gamma_{df_0} \cap \Gamma_{df_1}$.

Two ways to show compositions are quasi-isomorphic:

1) hybrid moduli spaces
2) Show {holomorphic disks} $\cong$ {Morse flow trees} (FO)

 $\cong$