Agnes Gadbled - Exact Fukaya category

Note Title 6/15/2011

(M, ω) symplectic mfd $\omega = d\theta$

(eg. $M = T^*X$)

Objects: compact exact Lagrangians: L s.t. $\theta|_L$ exact.

Morphisms: intersections

A_∞ structures: μ^d

If $L_0 \pitchfork L_1$ then $CF^*(L_0, L_1) := k\langle L_0 \cap L_1 \rangle$

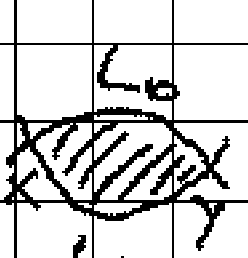
char$(k) = 2$

char$(k) \neq 2$ $\Rightarrow$ need extra structure on Lags.

graded version: " " " " " .

$\mu^1 : CF(L_0, L_1) \to CF(L_0, L_1)$

$(\mu^1)^2 = 0$

$\mu^1(x) = \sum_y \#\Big($ J-hol strips  $\Big)/\mathbb{R}$ y

where J = almost-cx. structure compatible with ω.

A J-holomorphic strip is, more precisely,

$u : [0,1] \times \mathbb{R} \to M$

$\bar\partial_J u = 0$, i.e. $\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$

$u(0,s) \in L_0$

$u(1,s) \in L_1$

$\lim_{s \to +\infty} u(t,s) = y$

$\lim_{s \to -\infty} u(t,s) = x$

$E(u) = \int u^*\omega$ = bounded.

The moduli space of J-hol strips mod translation is a manifold (transversality) and the 0-dimensional component is compact (Gromov compactness).

The local dimension is given by Maslov index:

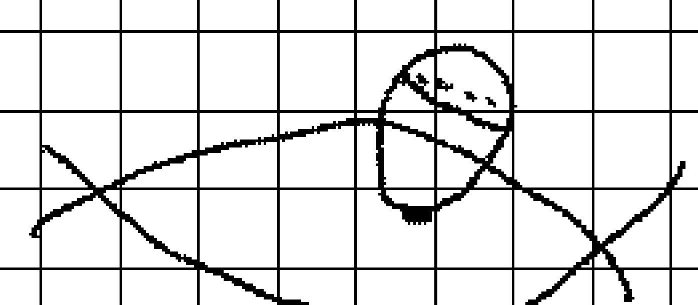
(V, ω) = symplectic vector space

$Gr(V)$ = Lag. Grassmannian

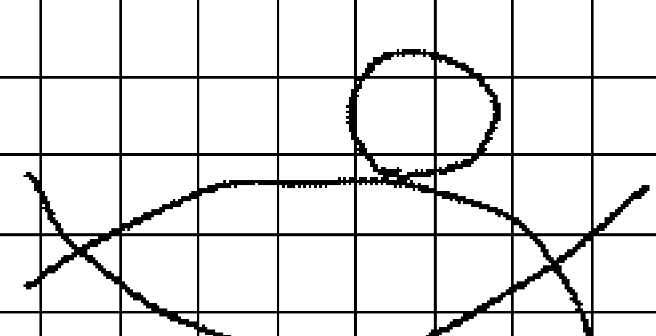
$\pi_1(Gr(V)) \cong \mathbb{Z}$

Given a hol. strip $u: Z \to M$ with boundary on L_0, L_1, trivialise $u^* TM$ over Z so that TL_0 is constant, then look at how TL_1 moves along the boundary: the Maslov index gives the index of ~~the~~ (the linearisation of) $\bar{\partial}_J$.

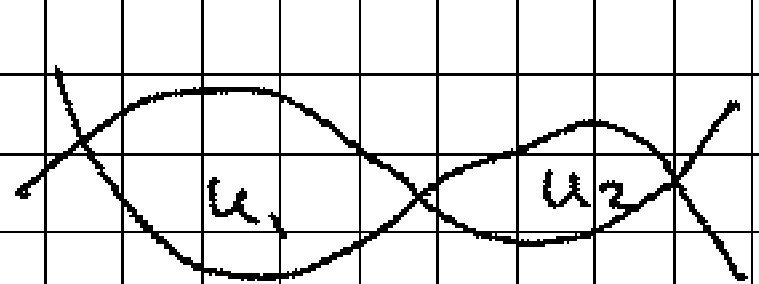
Compactness: given a sequence of strips, there is a subsequence converging to one of:



(doesn't happen: M exact)

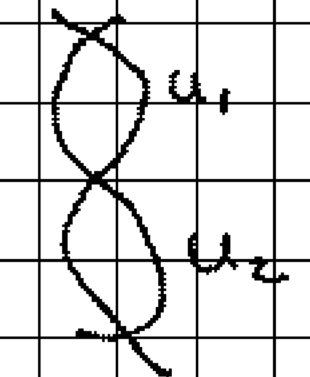


(doesn't happen: L exact)



$\sum \operatorname{ind}(u_i) = \operatorname{ind}(u)$

To show $(\mu^1)^2 = 0$, look at the 1-dimensional component of the moduli space of strips. It is compact with boundary given by



So the sum of boundary components is 0
$\Rightarrow (\mu^1)^2 = 0$.

It is always 0 if k has char $= 2$, if char $k \neq 2$, then we need to orient our moduli spaces of solutions. For this we need to consider Lagrangian <u>branes</u>:

- equip M with $\eta \in \Omega^{n,0}(TM)^{\otimes 2}$
- equip L with a <u>grading</u> $\alpha^{\#}$: a lift

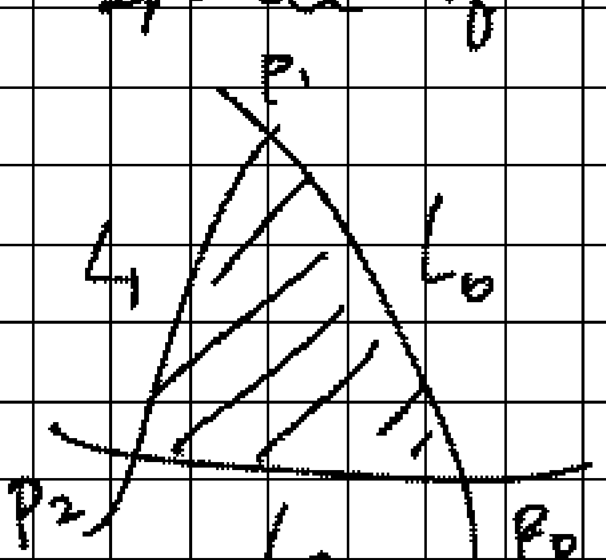
$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ L \arrow[ur, "\alpha^{\#}"] \arrow[r, "{\eta(TL,TL)}"'] & S^1 \end{tikzcd}$$

- fix a Pin structure on L.

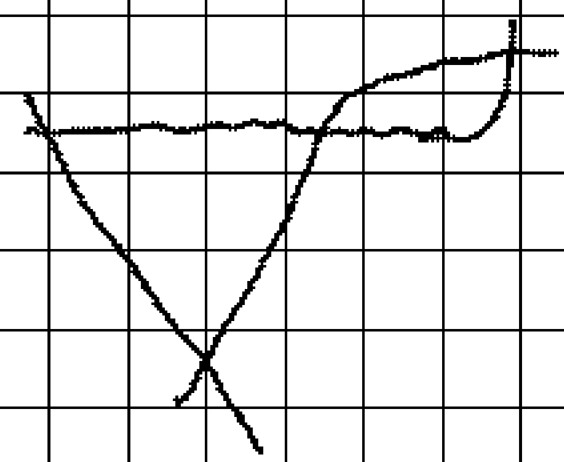
NB: If L_0, L_1 are not transverse, we must perturb one of them by a Hamiltonian isotopy so they are.

$$\mu^2 : \underset{p_2}{CF(L_1, L_2)} \otimes \underset{p_1}{CF(L_0, L_1)} \longrightarrow \underset{p_0}{CF(L_0, L_2)}$$

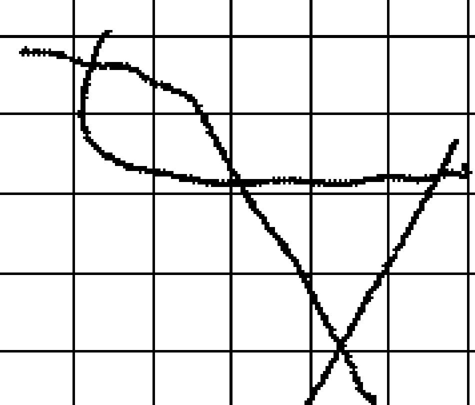
the coefficient of p_0 in $\mu^2(p_2, p_1)$ is the count of 0-dimensional part of the moduli space of J-holomorphic triangles



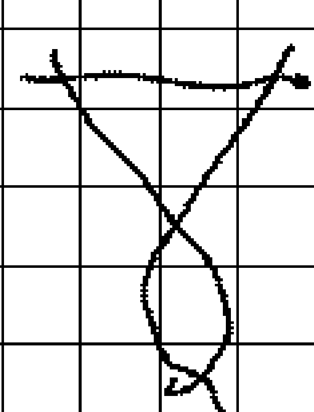
Looking at the boundary of the 1-dimensional part of the moduli space of triangles, it can break as follows:



$\mu^2(\mu^1(\cdot),\cdot)$



$\mu^2(\cdot,\mu^1(\cdot))$



$\mu^1(\mu^2(\cdot,\cdot))$

Again, the signed count is 0, so the Leibniz rule is satisfied.

PSS isomorphism

L compact exact

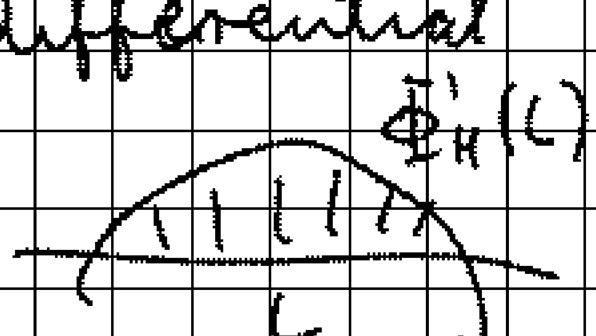
$CM(L)$ $\qquad\qquad$ $CF(L,L)$

$\uparrow$

generated by $L \pitchfork \Phi_H^1(L)$

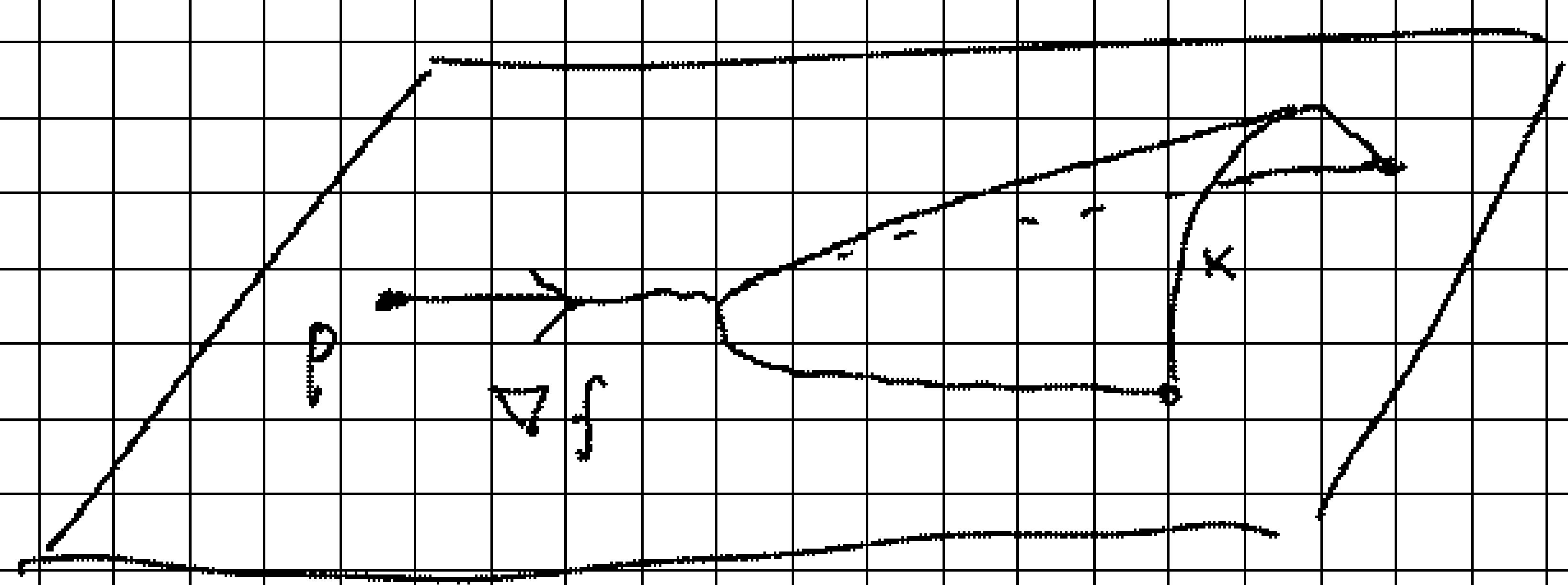
= Hamiltonian chords from L to L

differential

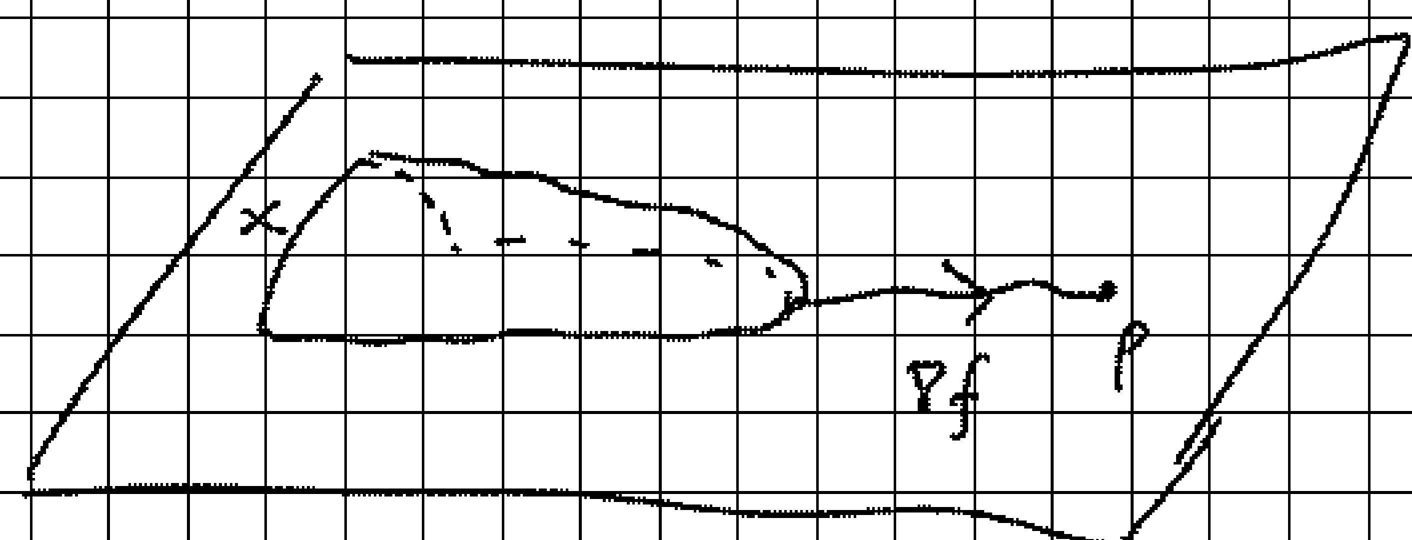


$\cong$ solutions of $\bar\partial_J + \nabla H = 0$.

$F: CM^*(L) \longrightarrow CF(L,L)$



$G: CF(L,L) \longrightarrow CM(L)$



Now we have:

$G \circ F$