Thomas Kragh -

Note Title 6/15/2011

$$\begin{tikzcd}
 & D^b_c(X) \arrow[rr] & & \mathrm{Fuk}(T^*X) \\
\mathrm{Tw\,Open}(X) \arrow[ur, "\sim"] & & & \mathrm{Mor}(X) \arrow[dl] \\
 & \mathrm{Open}(X) \arrow[ul] \arrow[uu] \arrow[r] & \mathrm{Open}_D(X) &
\end{tikzcd}$$

distributions

$f : X \to \mathbb{R}$ Morse function

g Riem. metric on X

$\mathrm{Crit}(f) = \{\text{crit points of } f\}$

$W^{\pm}(x) = \{w \in X \mid \lim_{t \to \pm\infty} \varphi_t(w) = x\}$

where φ_t is gradient flow of f.

$MC_*(X, f) = \mathbb{C}\langle \mathrm{Crit}(f) \rangle$

$d = m^1$ counts flow lines:

$d(x) = \sum_y \#(\text{flow lines from } x \text{ to } y)\, y$

Can now define A_∞ structure by counting gradient trees.

$\mathrm{Open}(X)$:

Objects: $\mathcal{U} = (U, m, \mathcal{L})$

$U \subset X$ open

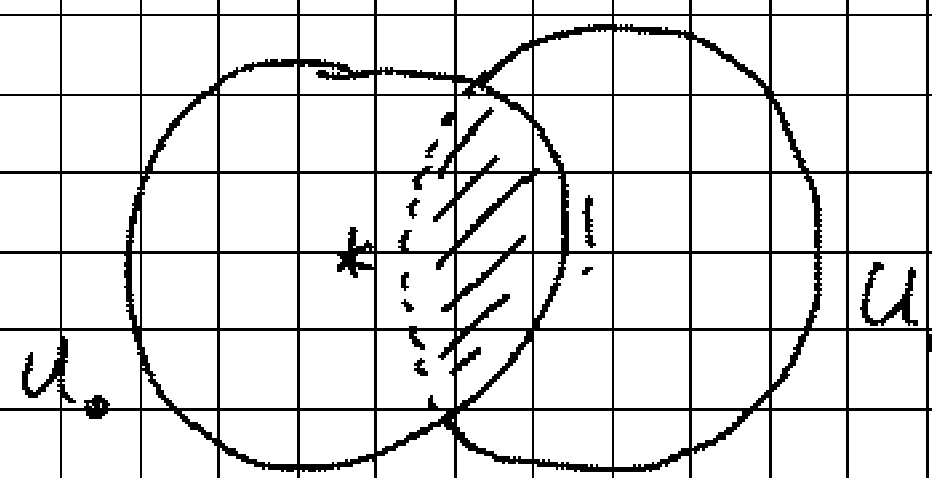
$m : X \to \mathbb{R}_{\geq 0}$ is a defining function, i.e.

$\{m = 0\} = X \setminus U$.

$\mathcal{L}$ is a local system on U.

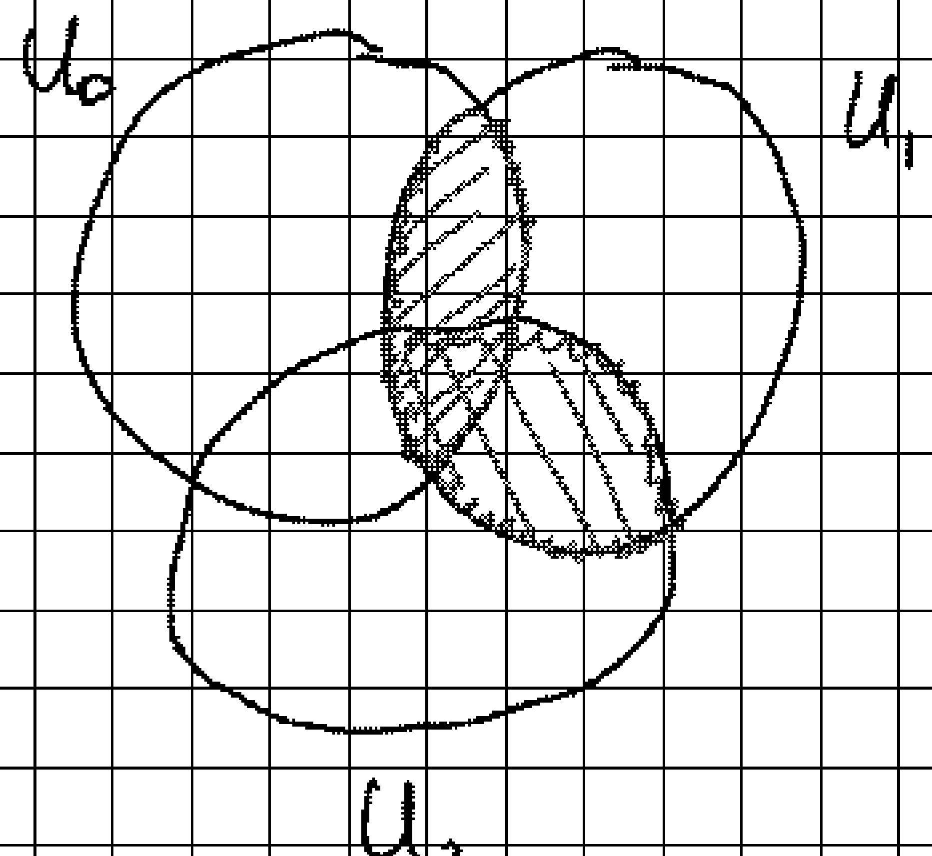
morphisms $(U_0, m_0, \mathcal{L}_0) \to (U_1, m_1, \mathcal{L}_1)$

is $\Omega^{\bullet}_{deR}(\overline{U_0} \cap U_1, (\partial U_0) \cap U_1; \mathcal{L}_0 \otimes \mathcal{L}_1^{\vee})$

= 

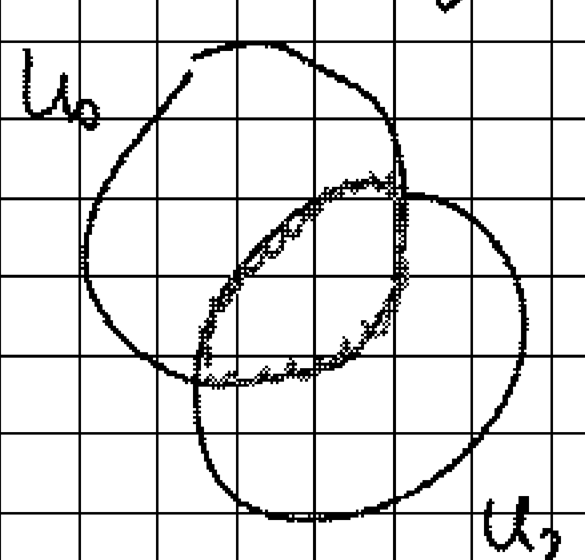
can write using !, *

composition: wedge product.

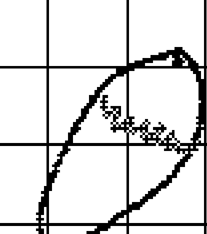


~~~~ = ! (forms vanish)
~~~~ = * (can approach

$\Big\downarrow$ wedge



Note: we can extend the form on $U_0 \cap U_1 \cap U_2$ by zero, ~~bcses~~ to $U_0 \cap U_2$, because it vanishes along



Functor to $D^b_c(X)$ is $p(U, m, \mathcal{L}) = j_* \mathcal{L}$.

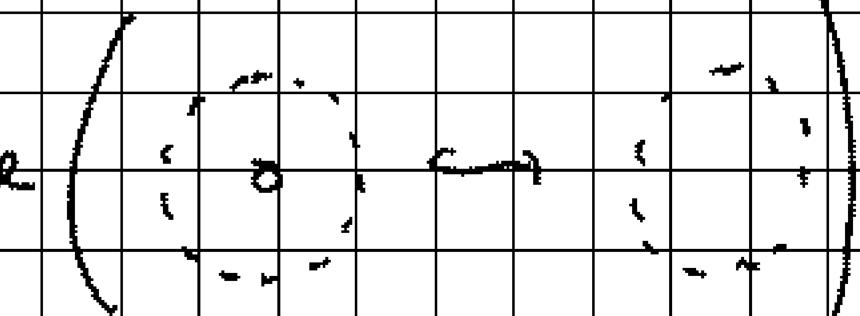
$\mathrm{Hom}(j_{0*}\mathcal{L}_0, j_{1*}\mathcal{L}_1) \cong \Omega^\bullet_{dR}(\overline{U}_0 \cap U_1, \partial U_0 \cap U_1; \mathcal{L})$.

(ex).

So we have defined a functor $P: \mathrm{Open}(X) \to D^b_c(X)$.

Its images generate: structure sheaves of opens generate constructible sheaves.

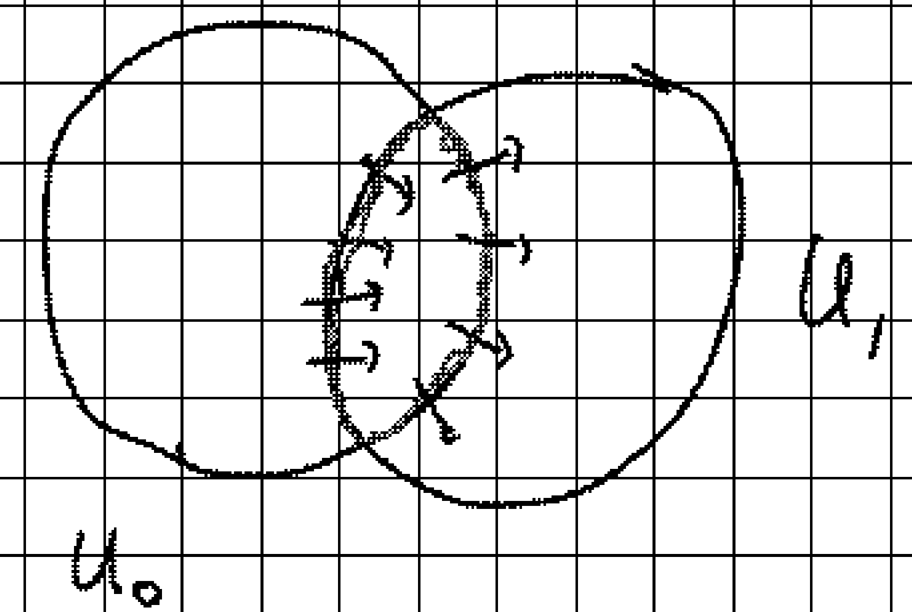
E.g. for a point:

$\bullet$ = cone 

Mor (X) is a new category:

Obj: same as Open(X)

Mor:

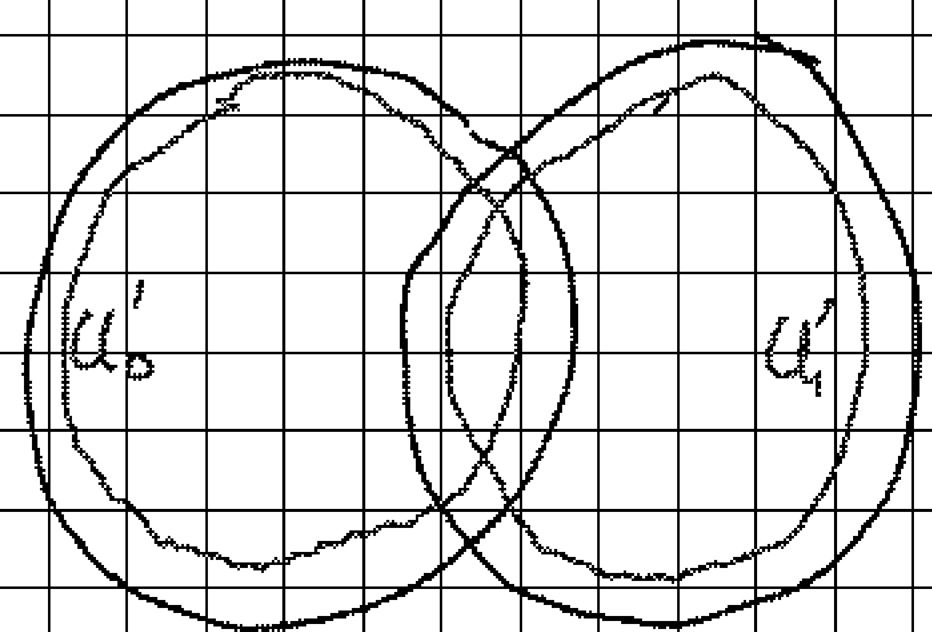


$H^*(\overline{\mathcal{U}}_0 \cap \mathcal{U}_1, \partial\mathcal{U}_0 \cap \mathcal{U}_1)$ is computed by Morse cohomology of a function going outwards along $\partial\mathcal{U}_0$, inwards along $\partial\mathcal{U}_1$.

Thm isotopy lemma:

replace $\mathcal{U}_0, \mathcal{U}_1$ by $\underbrace{\{m_0 > \eta_0\}}_{= \mathcal{U}_0'}$, $\underbrace{\{m_1 > \eta_1\}}_{= \mathcal{U}_1'}$

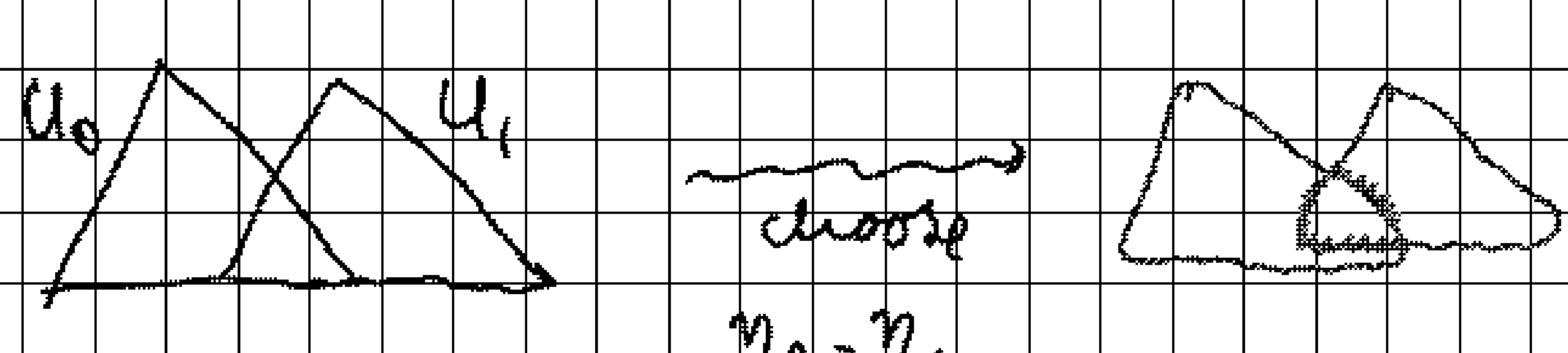
so we get



$f_0 = \log m_0$
$f_1 = \log m_1$

By choosing η_0, η_1 carefully, we get $\mathcal{U}_0' \cap \mathcal{U}_1'$ with a boundary with 2 components $\partial\mathcal{U}_0', \partial\mathcal{U}_1'$. Then $f_1 - \varepsilon_0 f_0$ points in along $\partial\mathcal{U}_0'$, out along $\partial\mathcal{U}_1'$.

E.g.



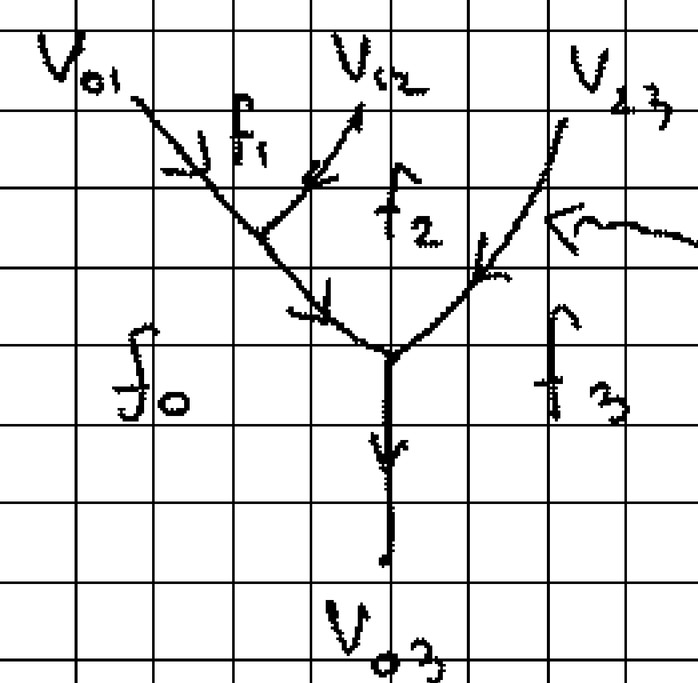
$$\mathrm{Hom}\big((U_0, m_0), (U_1, m_1)\big) = CM^*(f_{01})$$

with Morse differential.

Now define

$$\mu^d : \mathrm{Hom}\big((U_0,m_0),(U_1,m_1)\big) \otimes \dots \otimes \mathrm{Hom}\big((U_{d-1},m_{d-1}),(U_d,m_d)\big)$$
$$\longrightarrow \mathrm{Hom}\big((U_0,m_0),(U_d,m_d)\big)$$

by counting gradient flow trees:



gradient flowline of f_{23} $(= f_2 - \varepsilon_3 f_3)$.

Thm: $\mathrm{Mor}(X) \longrightarrow \mathrm{Open}(X)$ is an A_∞ equivalence.