Nadler - discussion

Note Title 6/14/2011

$f : X \longrightarrow Y$ subanalytic

6 operations (Groth.) + Verdier duality

$f^*, f_* \quad f^!, f_! \quad \otimes, \mathrm{Hom} \quad \mathbb{D}$

$f^! = \mathbb{D} f^* \mathbb{D}$

$f_! = \mathbb{D} f_* \mathbb{D}$

$\otimes = \Delta^* (\cdot \boxtimes \cdot)$

How to think about derived functors: <u>always</u> think about complexes, replace by resolutions.

<u>Resolving sheaves:</u>

I. Canonical resolutions

Model situation

A alg/k

$$\begin{tikzcd} \pi^* : k\text{-mod} \arrow[r, shift left] & A\text{-mod} : \pi_* \arrow[l, shift left] \end{tikzcd}$$

$\pi^*(V) = A \otimes_k V$ $\quad$ π_* = forget

$$\cdots \longrightarrow \underset{A \otimes_k M}{\overset{}{\pi^* \pi_* M}} \longrightarrow M$$

bar resolution of M

adjunction

<u>Godement resolution</u>

$p : X_{disc} \longrightarrow X$

discrete topology

$$p^* : Sh(X) \rightleftarrows Sh(X_{disc}) : p_*$$

$$\mathcal{F} \hookrightarrow p_* p^* \mathcal{F} \longrightarrow \cdots$$

= Godement resolution

(no good for computation)

II. Examples

(resolutions which are good for computations)

k_X = const. sheaf.

$k_X \hookrightarrow C^0_X \to C^1_X \to \cdots$ singular cochain sheaf

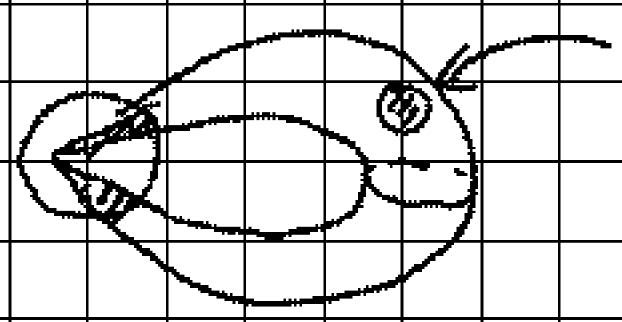
$k_X \hookrightarrow \Omega^0_X \to \Omega^1_X \to \cdots$ (X smooth)

ω_X = dualising sheaf

$\omega_X \hookrightarrow C^{BM}_{0[\dim X]} \to C^{BM}_{1[\dim X]} \to \cdots$

BM = Borel-Moore.

E.g.

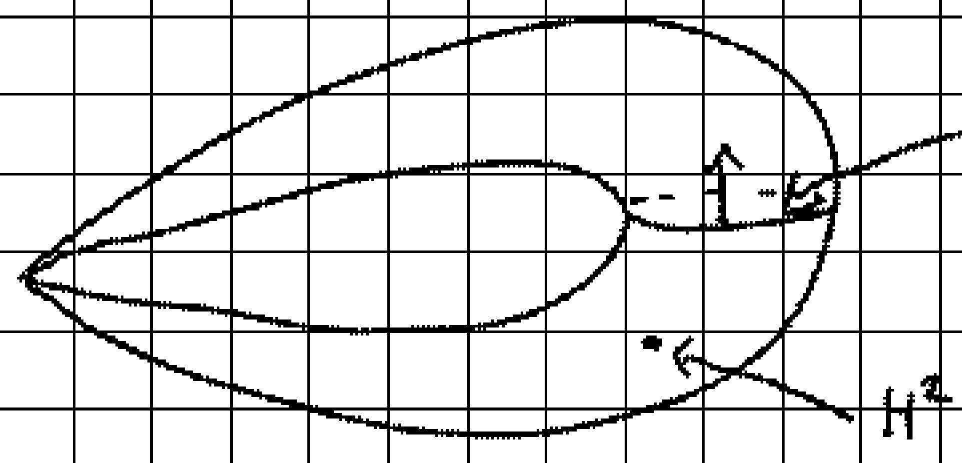


$C^0_X(U)$: singular cochains on U

$H^0 = k$

$H^1 = k$

$H^2 = k$



H^1 = "PD to this guy"

H^0

$H^*(\omega_X) = H_*(X)$

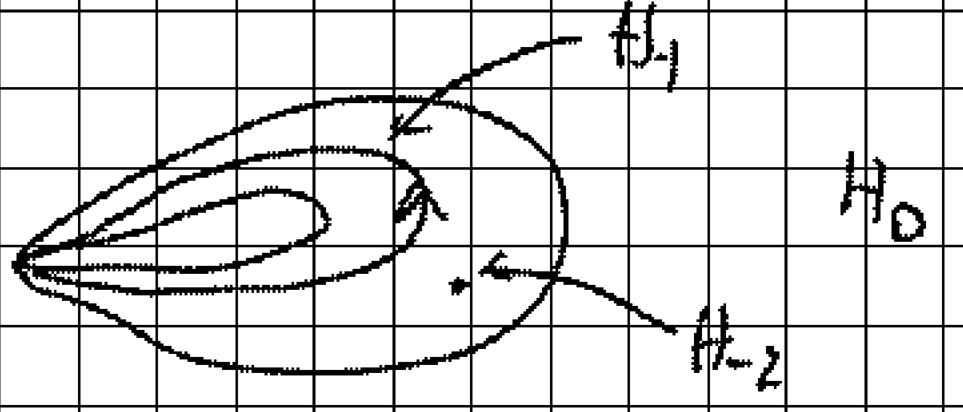
$= \Gamma(\omega_X)$

$= H^0 = k$

$H^{-1} = k$

$H^{-2} = k$

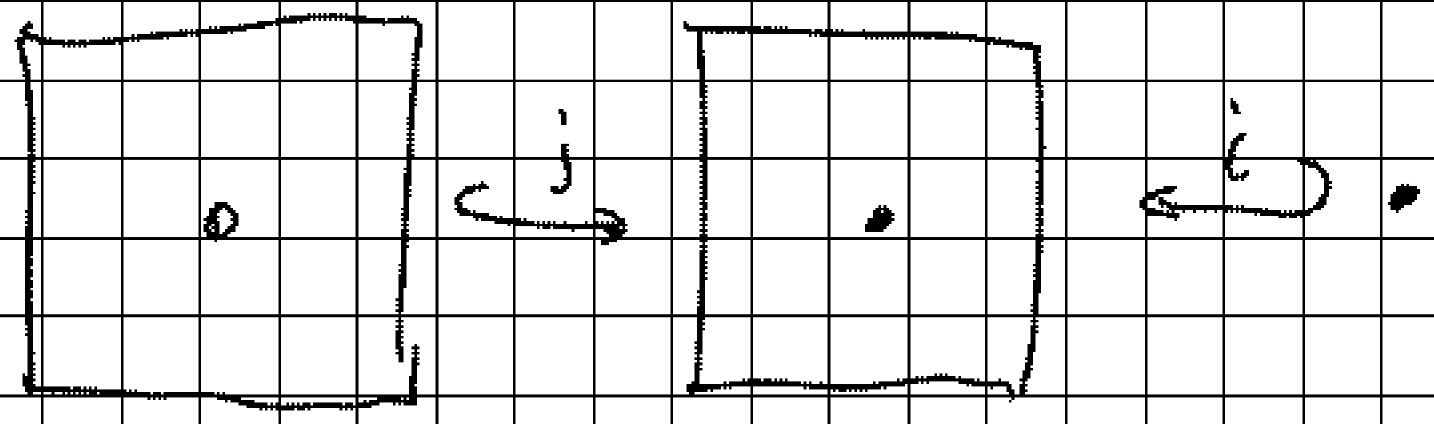
$=$



$D^b_c(X)$ = dg cat of const. on X

$\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ = complex (resolve $\mathcal{F}, \mathcal{G}$, take total complex)

E.g. $j : \mathbb{C}^* \hookrightarrow \mathbb{C} \hookleftarrow pt : i$



$f : X \to Y$

If f is proper, $f_! \xrightarrow{\sim} f_*$

If f is a submersion, oriented of codim k

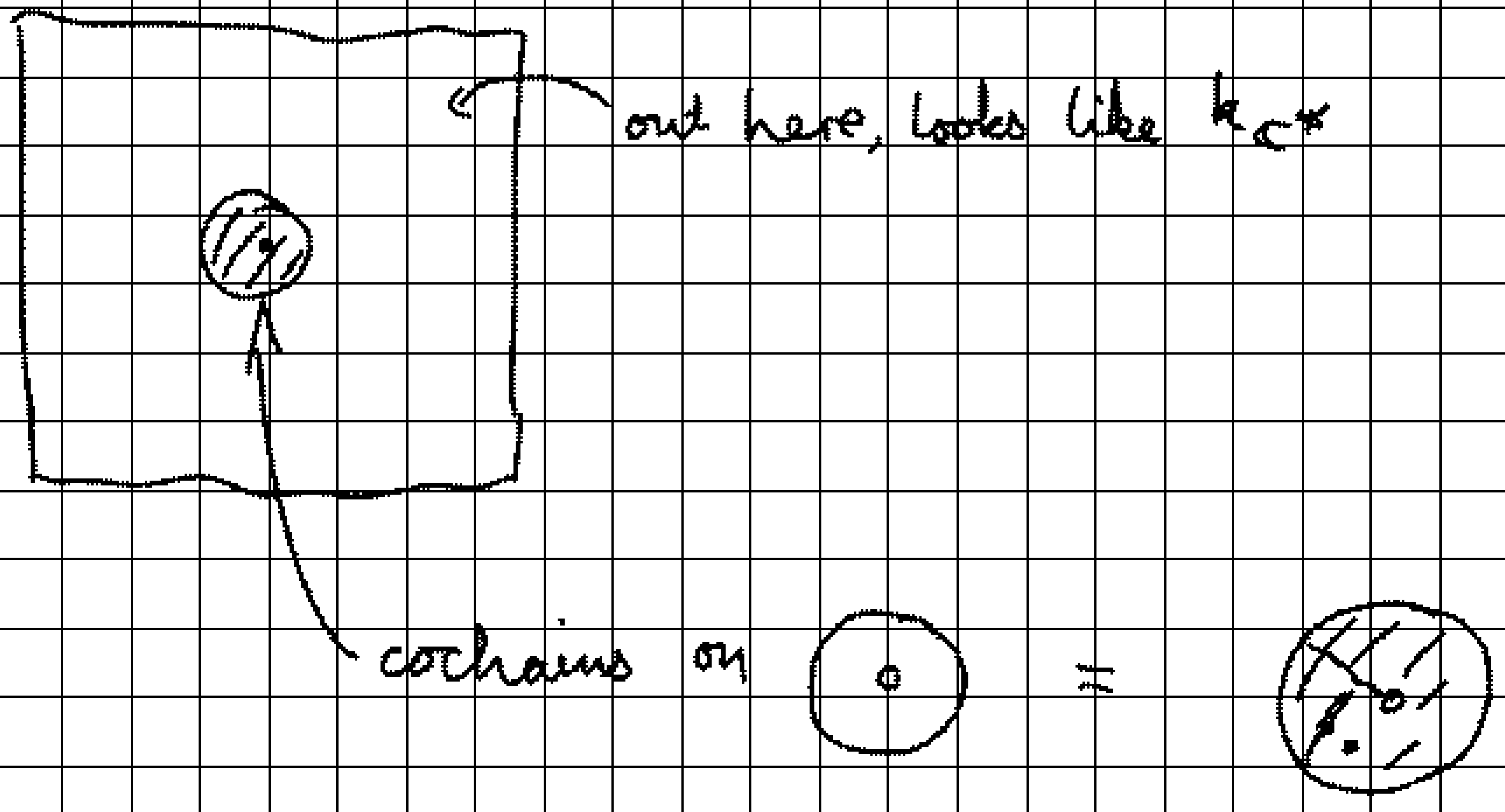
$\Rightarrow f^* = f^![-k]$

j is open $\Rightarrow$ smooth

i is closed $\Rightarrow$ proper

$j^*\mathcal{F} = \mathcal{F}|_{\mathbb{C}^*}$ (j^* exact $\Rightarrow$ don't need to resolve)

$j_* k_{\mathbb{C}^*} = j_* C^\bullet_{\mathbb{C}^*}$ $\boxed{\text{terminal}}$



We should get maps $\mathcal{F} \to j_* j^* \mathcal{F}$ and $j^* j_* \mathcal{F} \to \mathcal{F}_{\mathbb{C}^*}$ by adjunction

$$\begin{tikzcd} j^* j_* \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F}_{\mathbb{C}^*} \\ \mathcal{F}_{\mathbb{C}^*} \arrow[ur, "id"'] & \end{tikzcd}$$

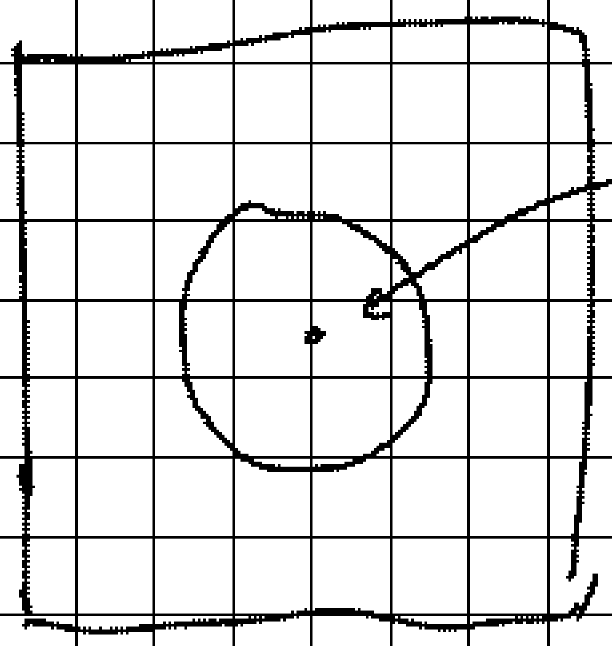
$$\begin{tikzcd} k_{\mathbb{C}} \arrow[r] \arrow[d, equal] & j_* j^* k_{\mathbb{C}} \arrow[d, equal] \\ C^\bullet_{\mathbb{C}} \arrow[r, "restriction"'] & C^\bullet_{\mathbb{C} \setminus \{0\}} \end{tikzcd}$$

$(j_!, j^!)$

- $j^! \simeq j^*$ restriction

- $j_! k_{\mathbb{C}^*} = j_! C^\cdot_{\mathbb{C}^*}$

  $\boxed{\text{initial}}$



cochains on pair $\left(▨ , \bullet \right) = (D^2, 0)$

(i.e. they vanish on $\bullet$, i.e. stay away from them)

$$\begin{tikzcd}
j_! j^! k_{\mathbb{C}} \arrow[r] \arrow[d, equal] & k_{\mathbb{C}} \arrow[d, equal] \\
C^\cdot_{(\mathbb{C}, pt)} \arrow[r] & C^\cdot_{\mathbb{C}}
\end{tikzcd}$$

$i^* \mathcal{F}$ = stalk at pt $(= \Gamma(\mathcal{U}, \mathcal{F})$, $\mathcal{U}$ = small nbhd of pt$)$

$i^! \mathcal{F}$ = costalk at pt $(= \Gamma((\mathcal{U}, \partial\mathcal{U}), \mathcal{F})$, sections over $\mathcal{U}$, rel boundary$)$

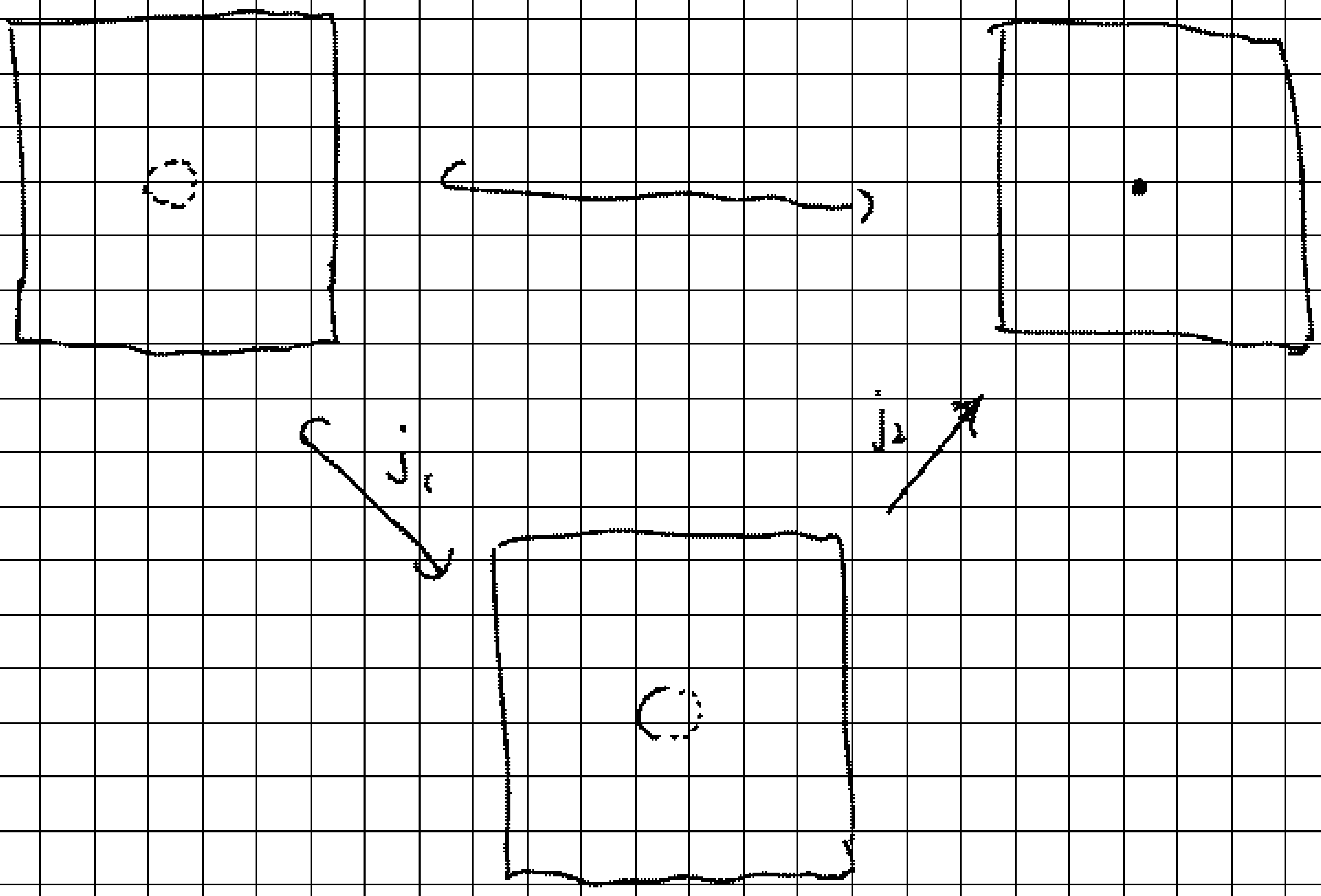
E.g. $\mathcal{F} = k_{\mathbb{C}}$

$i^* k_{\mathbb{C}} = k_{pt}$

$i^! k_{\mathbb{C}} = i^! C^\cdot_X = C^\cdot(\mathcal{U}, \partial\mathcal{U}) = k_{pt}$ in deg 2

$j_!$ = extension by 0

$\Rightarrow i^* j_! k_{\mathbb{C}^*} = 0$:



$T = j_{2*} \, j_{1!} \, k_{\mathbb{C}^*}$ tilting sheaf