

Sarah - Riemann-Hilbert Correspondence

Note Title

6/14/2011

$X = \mathbb{C}$ -variety, smooth

$D_X =$ sheaf of regular diff operators on X
(holom)

$\mathcal{O}_X =$ sheaf of algebras contained in D_X

T^*X

$\downarrow \pi$

X

$\text{Gr } D_X = \pi_* \mathcal{O}_{T^*X}$ (Gr = assoc. graded w.r.t. order of operator)

On A^n , $D_X = \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$

$$[\partial_i, x_j] = \delta_{ij}$$

$$\mathcal{O}_{T^*X} = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$$

$$\text{So } \partial_i \mapsto \xi_i$$

Defn: A D_X -module on X is a sheaf of (left) D_X -modules
(such that it is quasicohherent ~~as~~ as an \mathcal{O}_X -module).

Defn: A D_X -module is coherent if it locally has a presentation by finitely many copies of D_X .

NB: coherent as D_X -module $\not\Rightarrow$ coherent as an \mathcal{O}_X -mod.

e.g. D_X is not a coherent \mathcal{O}_X -module.

Thm: A D_X -mod is coherent as an \mathcal{O}_X -module iff
it is a vector bundle w/ flat connection.

NB: connection $\rightsquigarrow \nabla_j$ acts as ∇_j
flat $\rightsquigarrow [\nabla_i, \nabla_j] = 0$.

Riemann-Hilbert correspondence

$$D_{\text{rh}}^b(D_X) \cong D_c^b(X)$$

\uparrow

"regular holonomic"

NB: \mathbb{C} -c \rightarrow strata are \mathbb{C} -manifolds

$$\text{NB: } D_{\text{rh}}^b(D_X) \supseteq \text{RH}(D_X)$$

↑
corresponds to "perverse sheaves"
 $\text{Perv}(X)$

"H": holonomic D-modules

For a coherent D-module M , can locally find a filtration F s.t. $\text{Gr}^F M$ is coh. $\text{Gr} D_X$ -module

$$\text{Gr} D_X = \pi_* \mathcal{O}_{T^*X}$$

$$\pi^* \text{Gr}^F M = \mathcal{O}_{T^*X}\text{-module}$$

$$I_M = \sqrt{\text{Ann}_{\mathcal{O}_{T^*X}} \pi^* \text{Gr}^F M} \quad (\text{radical})$$

$$\text{Ch}(M) = V(I_M) = \text{characteristic variety of } M$$

(= "support of M ")

Properties of $\text{Ch}(M)$

- (1) $\dim \text{Ch}(M) \geq \dim X$ if $M \neq 0$
- (2) $\text{Ch}(M)$ is a conical coisotropic subvariety
- (3) $\pi(\text{Ch}(M)) = \text{supp}(M)$

Defn: M is holonomic if $M=0$ or $\dim \text{Ch}(M) = \dim X$
(i.e. $\text{Ch}(M)$ is Lagrangian)

Picture: $T^*X = \text{Spec } \mathcal{O}_{T^*X}$

" $\text{Spec } D_X$ " is a fuzzy version of T^*X .

A D_X module has some support, which is fuzzy.

We dilate, by \mathbb{R}_+ action on T^*X , to get a conical version of the support.

\mathbb{R}_+ action \leftrightarrow grading F on M .

If we were in affine space it would just be

the \mathcal{O}_{T^*X} -module $\text{Gr}^F M$, and the corresponding subvariety.

E.g. (1) $\mathcal{O}_X \rightsquigarrow \text{Ch}(\mathcal{O}_X) = X \subset T^*X$

(2) $D_X \rightsquigarrow \text{Ch}(D_X) = T^*X$

$\Rightarrow D_X$ is not holonomic

So free D -modules are not in the category of holon D -modules.

Thm: $D^b(\text{Hol}(X)) \cong D^b_k(D_X)$

\uparrow D_X -complexes with holonomic cohomology do not give you anything new.

In contrast, $D^b(\text{Sh}_{\text{a-c}}(X)) \not\cong D^b_{\text{a-c}}(X)$.

Thm: For M a hol. D_X -module, $\exists U \subset \text{supp} M$ open dense s.t. $M|_U$ is a vector bundle with flat connection.

"Cor": M hol. D_X -module \Rightarrow can find a stratification of X s.t. $M|_S$ is a v.b. w/ connection.

$D^b_k(X) \xrightarrow{\text{sol}} D^b_{\text{a-c}}(X)$

NB. essentially surjective but not fully faithful.

$\text{Sol}(M^*) = \text{Hom}^*_{D_X}(M^*, \mathcal{O}_X^{\text{an}})$

analytic

E.g. $X = \mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$

D_X -module $M = D_X / D_X(p)$

$p =$ linear diff eqn
 $p = \partial_x - x$

$\text{Hom}^*_{D_X}(M, \mathcal{O}_X^{\text{an}})$

$1 \mapsto f(x)$

\uparrow
generates
 D_X

\Rightarrow need $p(f) = 0$

$\Rightarrow f(x) = c e^{\frac{1}{2}x^2}$

$\Rightarrow \text{Sol}(M) = \text{loc const } \mathbb{C}\text{-sheaf gen. by the soln of } p(f) = 0$

$$D_n^b(X) \xrightarrow{\text{DR}} D_{\text{reg}}^b(X)$$

↖ De Rham

$$\text{DR}(M^*) = \underbrace{\Omega^*(D_x)}_{\cong \omega_x} \otimes_{D_x} M^*[d_x]$$

"R": regular singularities

$$A' \quad D_x = \mathbb{C}[x, \partial]$$

$$D_x / D_x p$$

$$p = x\partial - 1$$

$$p(f) = 0$$

$$\partial f = \frac{1}{x} f$$

↑ pole of order 1.

Moral: we've seen holonomic D-modules (~~on a curve, say~~) are vector bundles away from some lower-dim stratum. 'Regular' means the connection on our vector bundle has only poles of order ≤ 1 around these lower-dim strata.

E.g. $p = \partial_x - x$ is not regular: at ∞ , ($w = \frac{1}{x}$), it looks like $\partial f = \frac{1}{w^2} \dots$ something
 ↖ order > 1 .

In general:

M has reg. sing. if $J_M = \text{Ann Gr } M$
 (i.e. don't need to take radical).

Main theorem: (regular holonomic D-modules)

$$\downarrow \text{g.e.}$$

\mathcal{O} -constructible sheaves.

So Riemann - Hilbert says there is an equivalence

