Sarah - Riemann-Hilbert Correspondence

Note Title 6/14/2011

X = $\mathbb{C}$-variety, smooth

D_X = sheaf of regular diff operators on X
(holom)

$\mathcal{O}_X$ = sheaf of algebras contained in D_X

$$\begin{array}{c} T^*X \\ \big\downarrow \pi \\ X \end{array}$$

$\mathrm{Gr}\, D_X = \pi_* \mathcal{O}_{T^*X}$ (Gr = assoc. graded w.r.t. order of operator)

On $\mathbb{A}^n$, $D_X = \mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$

$$[\partial_i, x_j] = \delta_{ij}$$

$$\mathcal{O}_{T^*X} = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$$

So $\partial_i \mapsto \xi_i$

<u>Defn</u>: A D_X-module on X is a sheaf of (left) D_X-modules (such that it is quasicoherent ~~[illegible]~~ as an $\mathcal{O}_X$-module).

<u>Defn</u>: A D_X-module is coherent if it locally has a presentation by finitely many copies of D_X.

NB: coherent as D_X-module $\not\Rightarrow$ coherent as an $\mathcal{O}_X$-mod.

e.g. D_X is not a coherent $\mathcal{O}_X$-module.

<u>Thm</u>: A D_X-mod is coherent as an $\mathcal{O}_X$-module iff it is a vector bundle w/ flat connection.

NB: connection $\rightsquigarrow$ ∂_j acts as ∇_j

flat $\rightsquigarrow$ $[\nabla_i, \nabla_j] = 0$.

<u>Riemann-Hilbert correspondence</u>

$$D^b_{rh}(D_X) \cong D^b_c(X)$$

"regular holonomic"

NB: $\mathbb{C}$-c $\Rightarrow$ strata are $\mathbb{C}$-manifolds

NB: $D^b_{rh}(\mathcal{D}_X) \supseteq RH(\mathcal{D}_X)$

corresponds to "perverse sheaves"
$Perv(X)$

"H": holonomic D-modules

For a coherent D-module $\mathcal{M}$, can locally find a filtration F s.t. $Gr^F \mathcal{M}$ is coh. $Gr D_X$-module

$$Gr D_X = \pi_* \mathcal{O}_{T^*X}$$

$\pi^* Gr^F M$ - $\mathcal{O}_{T^*X}$-module

$$I_M = \sqrt{Ann_{\mathcal{O}_{T^*X}} \pi^* Gr^F M} \quad \text{(radical)}$$

$Ch(M) = V(I_M)$ = characteristic variety of M

(= "support of M")

<u>Properties of $Ch(M)$</u>

(1) $\dim Ch(M) \geq \dim X$ if $M \neq 0$

(2) $Ch(M)$ is a conical coisotropic subvariety

(3) $\pi(Ch(M)) = supp(M)$

<u>Defn</u>: M is holonomic if $M = 0$ or $\dim Ch(M) = \dim X$

(i.e. $Ch(M)$ is Lagrangian)

Picture: $T^*X = Spec\ \mathcal{O}_{T^*X}$

"Spec D_X" is a fuzzy version of T^*X.

A D_X module has some support, which is fuzzy.

We dilate, by $\mathbb{R}_+$ action on T^*X, to get a conical version of the support.

$\mathbb{R}_+$ action $\longleftrightarrow$ grading F on M.

If we were in affine space it would just be the $\mathcal{O}_{T^*X}$-module $Gr^F M$, and the corresponding subvariety.

E.g. (1) $\mathcal{O}_X \rightsquigarrow Ch(\mathcal{O}_X) = X \subset T^*X$

(2) $D_X \rightsquigarrow Ch(D_X) = T^*X$

$\Rightarrow D_X$ is not holonomic

So free D-modules are not in the category of holon D-modules.

Thm: $D^b(\mathrm{Hol}(X)) \cong D^b_h(D_X)$

D_X-complexes with holonomic cohomology do not give you anything new.

In contrast, $D^b(Sh_{\mathbb{C}\text{-}c}(X)) \not\cong D^b_{\mathbb{C}\text{-}c}(X)$.

Thm: For M a hol. D_X-module, $\exists\, U \subset \mathrm{supp} M$ open dense s.t. $\mathcal{M}|_U$ is a vector bundle with flat connection.

"Cor": $\mathcal{M}$ hol. D_X-module $\Rightarrow$ can find a stratification of X s.t. $\mathcal{M}|_S$ is a v.b. w/ connection.

$D^b_h(X) \xrightarrow{\mathrm{sol}} D^b_{\mathbb{C}\text{-}c}(X)$ NB. essentially surjective but not fully faithful.

$\mathrm{Sol}(\mathcal{M}^\bullet) = \underline{\mathrm{Hom}}^\bullet_{D_X}(\mathcal{M}^\bullet, \mathcal{O}_X^{an})$ analytic

E.g. $X = \mathbb{A}^1 = \mathrm{Spec}\, \mathbb{C}[t]$

D_X-module $M = D_X / D_X(p)$ p = linear diff eqn
$p = \partial_x - x$

$\underline{\mathrm{Hom}}^\bullet_{D_X}(M, \mathcal{O}_X^{an})$

$1 \mapsto f(x)$ $\Rightarrow$ need $p(f) = 0$

generates$/D_X$ $\Rightarrow f(x) = c\, e^{\frac{1}{2}x^2}$

$\Rightarrow \mathrm{Sol}(M) =$ loc const. $\mathbb{C}$-sheaf gen. by the soln of $p(f) = 0$.

$$D^b_h(X) \xrightarrow{DR} D^b_{\mathbb{C}\text{-}c}(X)$$

De Rham

$$DR(\mathcal{M}^\cdot) = \underbrace{\Omega^\cdot(D_X)}_{\simeq\, \omega_X} \otimes_{D_X} \mathcal{M}^\cdot [d_X]$$

"R": regular singularities

$\mathbb{A}^1$ $D_X = \mathbb{C}[x, \partial]$

$D_X / D_X\, p$ $\quad p = x\partial - 1$

$p(f) = 0$

$\partial f = \frac{1}{x} f$

pole of order 1.

Moral: we've seen holonomic D-modules ~~for a curve, say~~ are vector bundles away from some lower-dim stratum. 'Regular' means the connection on our vector bundle has only poles of order ≤ 1 around these lower-dim strata.

E.g. $p = \partial_x - x$ is not regular: at ∞, $(w = \frac{1}{x})$, it looks like $\partial f = \frac{1}{w^3} \cdots$ something

order > 1.

In general:

$\mathcal{M}$ has reg. sing. if $J_M = \mathrm{Ann}\, \mathrm{Gr}\, M$

(i.e. don't need to take radical).

Main theorem: (regular holonomic D-modules)

$\wr\!\!\downarrow$ q.e.

$\mathbb{C}$-constructible sheaves.

So Riemann-Hilbert says there is an equivalence

$$\begin{array}{c}\text{differential}\\ \text{equations}\\ \left(\begin{array}{c}\text{regular}\\ \text{holonomic}\end{array}\right)\end{array} \longleftrightarrow \text{their solutions}$$

need this condition for this to be an equivalence.