Faisal - intersections of Lagrangian cycles

Note Title 6/14/2011

$$\begin{tikzcd} K_0(D^b_c(X)) \arrow[r, "\overset{cc}{\sim}"] \arrow[dr, leftrightarrow, "\wr"] & L^+(X) \\ D^b_c(X) \arrow[u] & CF(X) \arrow[u, "\wr cc"'] \end{tikzcd}$$

$D^b_c(X) =$ dg category of constructible complexes (i.e., ones whose cohomology sheaves are constructible).

K_0 = if $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \xrightarrow{[1]}$ is an exact triangle, then

$[F_2] = [F_1] + [F_3]$

Goal: show that these maps respect some "natural" pairings $(\cdot, \cdot)$.

① On $K_0(D^b_c(X))$

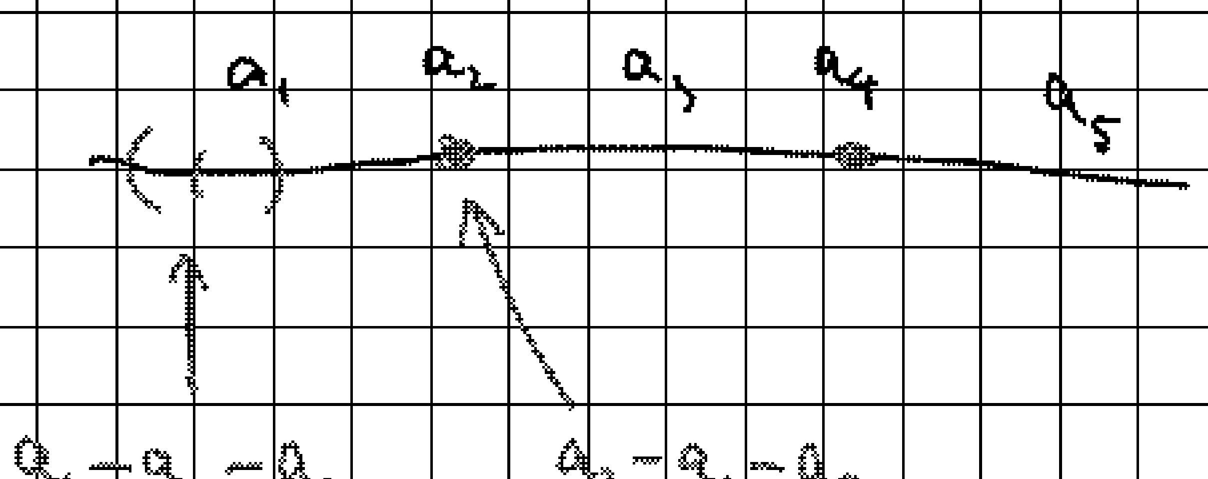
$$I(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = \sum_i (-1)^i \dim \operatorname{Ext}^i(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$

② On $CF(X)$

$$I(f, g) = \chi(Df \cdot g)$$

Df = Verdier dual of f

E.g. of calculating Df:



(Euler integral)

More formally, $Df(x) = \lim_{\varepsilon \to 0} \int f \, \mathbb{1}_{B(x,\varepsilon)} \, d\chi$

NB. $I(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = I\left(\chi(\mathcal{F}^\cdot), \chi(\mathcal{G}^\cdot)\right)$

③ $I(\cdot,\cdot)$ on $L^+(X)$

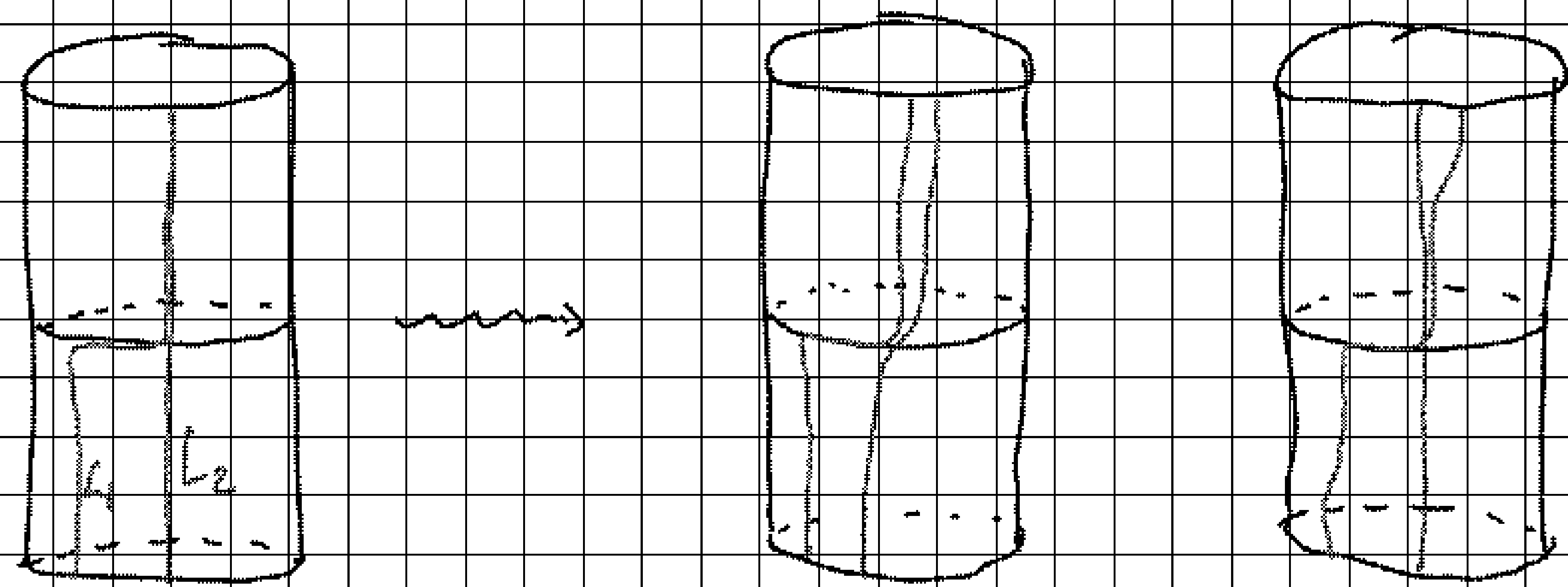
want it to be some kind of intersection number.

$I(L_1, L_2) := L_1 \cap \gamma_\varepsilon(L_2)$

where γ_ε = time-ε normalized geodesic flow

(from Hamiltonian $H(p) = |p|$ at ∞).

So, for example:



$I(L_1, L_2) = 0$ $\qquad$ $I(L_2, L_1) = 1$

NB: Using subanalyticity, we can perturb so that our Lag cycles intersect transversely at regular points (i.e. Λ_0 from Thomas' talk).

Global Index Formula

$I(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = I(CC(\mathcal{F}^\cdot), CC(\mathcal{G}^\cdot))$.

Special case: (X cpct oriented)

① $\mathcal{F} = k_X \Rightarrow CC(\mathcal{F}) = T^*_X X$ = 0-section

$\chi(X, \mathcal{G}) = T^*_X X \cap CC(\mathcal{G})$

Dubson - Kashiwara index formula.

② $i: Y \hookrightarrow X$ $\quad$ k-dim submfd of X (closed)

$\mathcal{G} = i_! \mathcal{L}$ $\quad$ ($\mathcal{L}$ = local system on Y)

$\Rightarrow CC(\mathcal{G}) = (-)^k \operatorname{rank}(\mathcal{L})\, T^*_Y X$

$\mathcal{F} = k_X$

$$\chi(X, i_*\mathcal{L}) = (-1)^k \operatorname{rank}(\mathcal{L})\; T_x^* X \cap T_Y^* X$$

$$\begin{aligned} &\shortparallel \\ &\chi(Y, \mathcal{L}) \\ &\shortparallel \\ &\chi(Y)\, \operatorname{rk}(\mathcal{L}) \end{aligned}$$

$$\Rightarrow \chi(Y) = (-1)^k\, T_x^* X \cap T_Y^* X.$$

③ Take $Y = X$ in ②

$$\chi(X) = (-1)^{\dim X}\, T_x^* X \cap T_x^* X$$

$$= (-1)^{\dim X}\, \varepsilon(T^* X) \cdot [X] \qquad \varepsilon = \text{Euler class}$$

$$= (-1)^{\dim X} (-1)^{\dim X}\, \varepsilon(X) \cdot [X]$$

$$= \varepsilon(X) \cdot [X]$$

Poincaré–Hopf index theorem.

E.g. $T^* S^1$, two constr. functions