Ali - cotangent bundles

Note Title 6/14/2011

Classical mechanics

M configuration space

$L : TM \longrightarrow \mathbb{R}$ Lagrangian

$\gamma : [t_0, t_1] \longrightarrow M$

$\gamma(t_0) = x_0 \quad \gamma(t_1) = x_1$

Principle of least action:

$$S(\gamma) = \int_{t_0}^{t_1} L(\gamma, \dot\gamma)\, dt$$

γ is a physical path if local minimum for S.

E.g. g a metric on M

$$L(x, v) = g_x(v, v)$$

Euler - Lagrange equations; for a physical path:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q}$$

E.g. geodesic flow on $TS^1 = S^1 \times \mathbb{R}$
(θ, λ)

$$L : S^1 \times \mathbb{R} \to \mathbb{R}$$
$$(\theta, \lambda) \longmapsto \lambda^2$$
$$\frac{\partial L}{\partial \lambda} = 2\lambda$$
$$\frac{d}{dt}(2\lambda(t)) = 0$$

Hamiltonian formulation:

$$\Phi : TM \longrightarrow T^*M$$
$$\underset{T_qM}{\overset{v}{\cap}} \longmapsto d\left(L|_{T_qM}\right)_v \in T_v^*(T_qM) \cong (T_qM)^*$$

Assume Φ is a diffeomorphism

$H : T^*M \longrightarrow \mathbb{R}$

$H(p) := dL_{\Phi^{-1}(p)}(\Phi^{-1}(p)) - L(\Phi^{-1}(p))$

E.g. Hamiltonian mechanics

T = kinetic energy

V = pot. energy

$L = \frac{1}{2}(T - V)$

$H = T - L = \frac{1}{2}(T + V)$

$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$

Now given T^*M, we define a canonical 1-form λ:

$$\lambda_{(q,p)} = -\pi^* p$$
$$= -\sum_i p_i \, dq^i$$

symplectic form $\quad d\lambda = \sum_i dq^i \wedge dp_i$

Hamiltonian differential equation:

given $H : T^*M \to \mathbb{R}$

$\iota_{X_H} \omega = dH$ defines X_H - Hamiltonian vector field

the diff. eqn. is the flow of X_H.

<u>Lagrangian submanifolds</u>:

- the zero section $M \subset T^*M$
- the graph of a closed 1-form
- the graph of an exact 1-form is an <u>exact</u> Lagrangian (i.e. $\lambda|_L$ is exact)
- Given a submanifold $N \hookrightarrow M$, there is an exact Lagrangian $T_N^*M \subset T^*M$ (the <u>conormal</u> of N)

$T_N^*M := \left\{ (q,p) \in T^*M : q \in N, \ p|_{T_qN} = 0 \right\}$

In local coordinates, if $N = (q_1, \dots, q_k, 0, \dots, 0)$

$$T_N^* M = (q_1, \dots, q_k, 0, \dots, 0, 0, \dots, 0, p_{k+1}, \dots, p_m)$$

<u>Hybrid Lagrangian</u>

$\alpha \in \Omega^1(N)$

$$T^*_{N,\alpha} M = T^*_N M + \alpha = \{(q,p) \in T^*M : p|_{T_qN} = \alpha\}$$

$T^*_{N,\alpha} M$ is Lagrangian $\Leftrightarrow$ α is a closed form on N

" " exact " $\Leftrightarrow$ " " exact " " "

<u>Exact symplectic manifold</u>

(M, ω, λ)

$d\lambda = \omega$

$\iota_Z \omega = \lambda$ $\quad Z$ = Liouville vector field

$\mathcal{L}_Z \omega = \omega \Rightarrow Z$ expands the symplectic form.

E.g. on T^*M, $Z = \sum p_i \frac{\partial}{\partial p_i}$.

<u>Defn</u>: contracting vector field: $\mathcal{L}_Z \omega = c(x)\omega \quad c(x) < 0$

($\Rightarrow \exists$ a primitive)

<u>Defn</u>: Weinstein mfd: (M, ω, Z, h)

where h is an exhausting Morse function

Z is contracting and complete

gradient-like: $dh(Z) > 0$ away from critical points

$Z = \nabla h$ near critical points

h has finitely many critical points.