Justin Curry - Constructible sheaves

Note Title 6/13/2011

Presheaves := Fun$(C^{op}, D)$

$C$ = Open$(X)$

very simple - just poset, morphisms $U \hookrightarrow V \hookrightarrow W$.

$D$ = Sets, Ab grps, $k$-modules, spaces, spectra, CAT

Sheaves: $U_i$ open cover of $U$

$$F(U) \longrightarrow \prod F(U_i) \longrightarrow \prod F(U_i \cap U_j)$$

exact.

Grothendieck: $Sh(X)$ (sheaves of abelian groups)
is an abelian category
(inherits properties of target category $D$).

Motto: constructible sheaves are glued together from

- flat vector bundles $(E, \nabla)$ $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$
$$\begin{array}{c} (E,\nabla) \\ \downarrow \\ M \end{array}$$
- local systems $d^\nabla : \Omega_r(E) \to \Omega_{r+1}(E)$, $(d^\nabla)^2 = 0$.

= representations of fundamental groupoid

$\rho : \pi_1(X, x_0) \longrightarrow \mathrm{Aut}(A)$ $A$ $k$-module.

- locally constant sheaf.

Defn: $\mathcal{F}$ over $X$ $\{U_i\}$ open cover

s.t. $\mathcal{F}|_{U_i} \cong \tilde{A}$ locally constant functions with values in $A$.

Complexes of sheaves

$$\mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2$$

$\rightsquigarrow \mathcal{H}^0(\mathcal{F}^\cdot), \mathcal{H}^1(\mathcal{F}^\cdot) \ldots$ cohomology sheaves

(NOT the same as sheaf cohomology)

i.e. $\mathcal{H}^i(\mathcal{F}^\cdot) = \overbrace{U \longmapsto H^i(U, \mathcal{F}^\cdot)}^{\text{sheafification}}$

Sheaf cohomology:

Take an injective resolution

$$\mathcal{F}^\cdot \longrightarrow I^\cdot$$

$$H^i(X, \mathcal{F}^\cdot) = H^i(\Gamma(X, I^\cdot)).$$

<u>Defn</u>: $\mathcal{F}^\cdot$ is <u>cohomologically locally constant</u> if each of the $\mathcal{H}^i(\mathcal{F}^\cdot)$ is loc. const.

<u>Defn</u>: $\mathcal{F}^\cdot$ is <u>weakly constructible</u> if $\exists$ partition $\{S^j\}$ of space $X$ s.t.

$$\mathcal{H}^i(\mathcal{F}^\cdot)\Big|_{S^j} \text{ is loc. const.}$$

<u>Defn</u>: If additionally $\mathcal{H}^i(\underset{\text{stalks}}{\mathcal{F}^\cdot_x})$ is a perfect complex (i.e., over a field, its f.d. vector space)

(NB $\mathcal{H}^i(\mathcal{F}^\cdot_x) \cong \mathcal{H}^i(\mathcal{F}^\cdot)_x$).

then it's constructible.

<u>Rmk</u>: Partition should be <u>tame</u>.

Schürmann - topological theory of singular spaces and constructible ...

Back to sheaves:

$f: X \longrightarrow Y$ continuous

$\mathcal{F} \in Sh(X)$ $\quad \mathcal{G} \in Sh(Y)$

$f_* \mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$.

$f^* \mathcal{G}(U) := \overbrace{\varinjlim_{V \supset f(U)} \mathcal{G}(V)}$

$$\left(f^* \mathcal{G}\right)_x \cong \mathcal{G}_{f(x)}.$$

<u>Thm:</u> $\mathrm{Hom}(f^* \mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_* \mathcal{F})$

E.g. $f : X \longrightarrow pt$

$$f_* \mathcal{F} = \Gamma(X, \mathcal{F}).$$

If $\mathcal{G} = k$ on the space pt

$\mathcal{F} \in Sh(X)$

$$\begin{array}{ccc} \mathrm{Hom}(f^* k, \mathcal{F}) & \cong & \mathrm{Hom}(k, f_* \mathcal{F}) \\ \wr\wr & & \wr\wr \\ \mathrm{Hom}(\tilde{k}, \mathcal{F}) & \cong & \mathrm{hom}(k, \Gamma(X, \mathcal{F})) \end{array}$$

sheaf of loc. const. maps to $k$.

Derive this adjunction: we get

$$Rf_* R\mathcal{H}om^\bullet(f^* \mathcal{G}^\bullet, \mathcal{F}^\bullet) \cong R\mathrm{Hom}^\bullet(\mathcal{G}^\bullet, Rf_* \mathcal{F}^\bullet)$$

$\mathcal{H}om$

$\mathcal{F}^\bullet \longrightarrow I^\bullet$

NB. $Rf_* \mathcal{F}^\bullet = f_* I^\bullet$

<u>Restricting to subspaces</u>

$j : Z \hookrightarrow X$ subspace top.

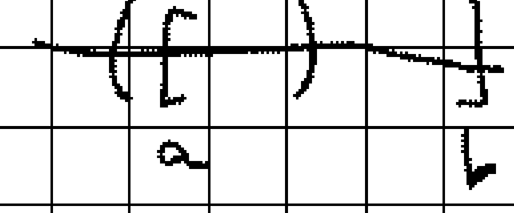
If $Z$ is open, $\mathcal{F}$ sheaf on $X$

$$\mathcal{F}|_Z \cong j^* \mathcal{F} \quad \text{sheaf on } Z$$

Want sheaves on $X$.

<u>Option #1:</u> <u>Defn:</u> $\mathcal{F}$ sheaf on $X$, $j : Z \hookrightarrow X$, $Z$ closed.

$$\mathcal{F}_Z = j_* j^* \mathcal{F}.$$

<u>Ex:</u> $k_{[a,b]}$  $k_{[a,b]}(U) = k$.

Defn: $f: X \to Y$ direct image with compact supports

$$f_! \mathcal{F}(V) = \{ s \in \Gamma(f^{-1}(V), \mathcal{F}) \mid \text{s.t. } f: \operatorname{supp}(s) \to V \text{ is proper} \}$$

$$\operatorname{supp}(s) = \{x \mid s_x \neq 0\} \text{ closed}$$

$$s_x = 0$$
$$s|_u.$$

Defn: $Z \hookrightarrow X$ loc. closed

(for closed sets $j_! = j_*$)

$Z = U \cap A$ (open, closed)

$$\mathcal{F}_Z = j_! j^* \mathcal{F}$$

Ex: $Z = (a,b)$ $\quad$ $\big(\overset{U}{\underset{a\quad b}{(\;\;)}}\big)$ $\quad$ $k_{(a,b)}(U) = 0.$

## Useful SES / Dist. triangles

$V$ closed in $X$

$U = X \setminus V$ $\qquad i: U \hookrightarrow X$

$\qquad\qquad\qquad j: V \hookrightarrow X$

$$\underset{H^*(X,V)}{i_! i^* \mathcal{F}} \to \underset{H^*(X)}{\mathcal{F}} \to \underset{H^*(V)}{j_* j^* \mathcal{F}} \xrightarrow{[1]}$$

Two open sets $U_1, U_2$, $\quad i_1, i_2: U_1, U_2 \hookrightarrow X$

$$0 \to \mathcal{F}_{U_1 \cap U_2} \to \mathcal{F}_{U_1} \oplus \mathcal{F}_{U_2} \to \mathcal{F}_{U_1 \cup U_2} \to \cdot \qquad \text{M-V}$$

Two closed sets $V_1, V_2$

$$0 \to \mathcal{F}_{V_1 \cup V_2} \to \mathcal{F}_{V_1} \oplus \mathcal{F}_{V_2} \to \mathcal{F}_{V_1 \cap V_2} \to \cdot$$

The first dist. triangle allows you to work inductively on ~~str~~ constr. sheaves ~~ove~~ w.r.t. some stratification $\rightsquigarrow$ shows as you go up strata you just get extensions.

Ex.:
$$\begin{tikzcd} R\Gamma_c(X, i_! i^* \mathcal{F}) \arrow[r] \arrow[d, phantom, "\cong"] & R\Gamma_c(X, \mathcal{F}) \arrow[d] \\ R\Gamma_c(U, i^*\mathcal{F}) & R\Gamma_c(X, j_* j^* \mathcal{F}) \arrow[ul, "{[1]}"'] \arrow[d, phantom, "\cong"] \\ & R\Gamma_c(V, j^*\mathcal{F}) \end{tikzcd}$$

$$H_c^\cdot(U, \mathcal{F}) \to H_c^\cdot(X, \mathcal{F}) \to H_c^\cdot(X \setminus U, \mathcal{F})$$

$$\chi_c(X, \mathcal{F}) = \chi_c(U, \mathcal{F}) + \chi_c(X \setminus U, \mathcal{F})$$

compactly supported $\chi$ is a measure.

Observation: $\mathcal{F}$ constr. sheaf

$$\chi(\mathcal{F})(x) = \sum (-1)^i \dim \mathcal{H}^i(\mathcal{F}^\cdot)_x = h(x)$$

locally const $\mathbb{Z}$-valued function on each stratum of the base space.

$$\chi_c(X, \mathcal{F}) = \sum_{\text{strata } S} \chi_c(S)\, h(x) = \int h \, d\chi$$

$x \in S$ — Euler integral (MacPherson)

$$CF(X, \mathbb{Z}) = \{ h : X \to \mathbb{Z} \mid h^{-1}(4) \text{ is tame} \}$$

constructible functions

$\cong$

$K_0(D_c^b(X))$.

E.g.
$$\begin{tikzcd} D^b_{\mathbb{R}\text{-}c}(X) \arrow[r, "f_!"] \arrow[d, "\chi"] & D^b_{\mathbb{R}\text{-}c}(Y) \arrow[r, "p_!"] \arrow[d, "\chi"] & D^b_{\mathbb{R}\text{-}c}(pt) \arrow[d, "\chi"] \\ CF(X) \arrow[r, "f_!"'] & CF(Y) \arrow[r] & CF(pt) \end{tikzcd}$$

real constr.

$$\int h \, d\chi = \int_Y \int_{f^{-1}(y)} h \, d\chi(x) \, d\chi(y)$$

<u>Thm</u>: $\mathrm{Hom}(Rf_! \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f^! \mathcal{G})$

Verdier Duality: $\exists\ f^! : D^b(Y) \longrightarrow D^b(X)$.

E.g. $p : X \longrightarrow pt$

$p^! k = \omega_X$ dualising complex

$D\mathcal{F} = \mathcal{H}om(\mathcal{F}, \omega_X)$.

$\chi_c(X, \mathcal{F}) = \chi(X, D\mathcal{F})$.

$\omega_X$ assigns <u>chains</u> on an open set $\mathcal{U}$.

$X$ is a manifold $\Rightarrow \omega_X = k[n]$ $(n = \dim X)$

$\Rightarrow$ Poincaré duality.

<u>Ex</u>: Take Hopf fibration $f : S^3 \to S^2$, what is derived pushforward of const sheaf?