Daniel Rowe - Examples

Note Title 6/13/2011

Unstratified space X, $\mathcal{F}$ loc. syst. on X.

$$\left[x \to y \right] \longleftrightarrow \left[F_x \xrightarrow{\sim} F_y \right]$$

homotopy
class of paths

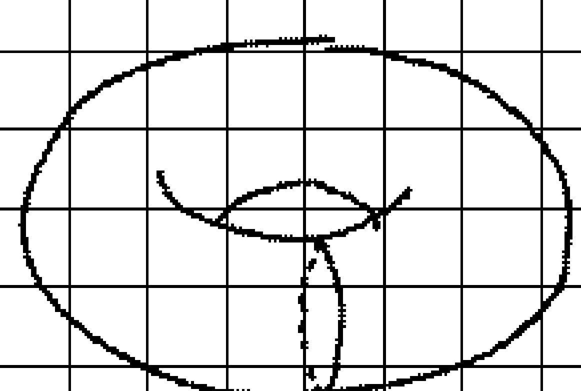
$$Loc(X) \longleftrightarrow Rep(\Pi_1 X) \longleftrightarrow Rep(\pi_1(X))$$

fund. groupoid

Fix stratification $\mathcal{S}$, $X = \coprod X_\alpha$

$F \in Sh_{c,\mathcal{S}}(X) \Leftrightarrow F|_{X_\alpha}$ is a local system $\forall\ \alpha$

E.g.



Study not <u>all</u> paths, but <u>exit paths</u>

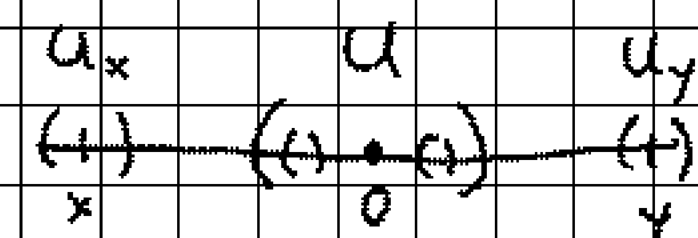
$EP(X, \mathcal{S})$ = <u>category with objects = paths</u>
<u>morphisms = paths that only move from</u>
<u>lower to higher-dimensional strata</u>

$$\left\{ \begin{matrix} \mathcal{S}\text{-constructible} \\ \text{sheaves on } X \end{matrix} \right\} \longleftrightarrow \left\{ \text{Representations of } EP(X, \mathcal{S}) \right\}$$

($\rightsquigarrow$ A_∞ functors, with higher homotopies of exit paths, if you talk about complexes of sheaves)

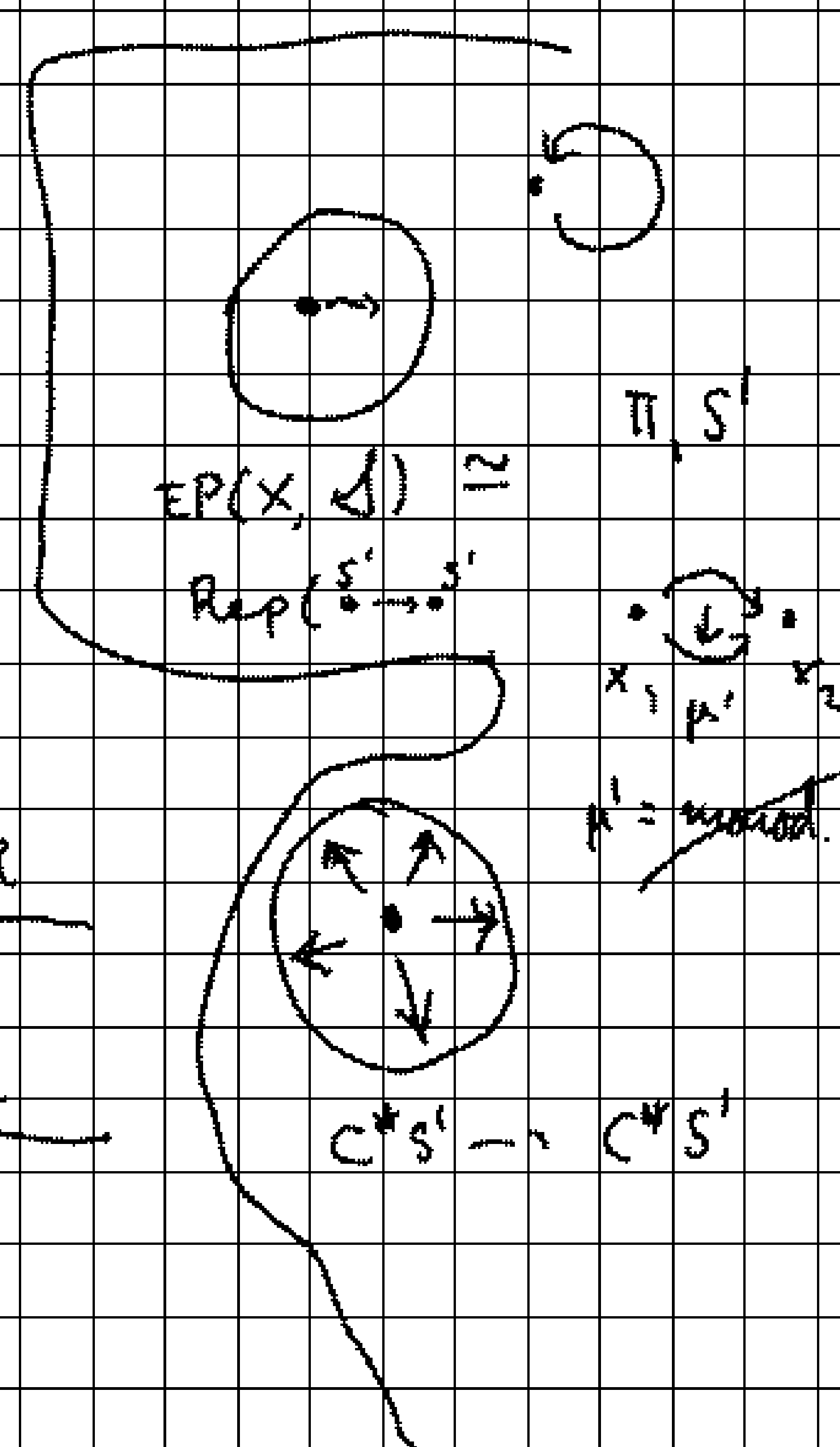
Eg: $X = \mathbb{R}$, $S = 0 \subset \mathbb{R}$

$F \in Sh_c(X)$



$$\begin{tikzcd} & F_0 = F(U) \arrow[dl] \arrow[dr] & \\ F(U_x) = F_x & & F_y = F(U_y) \end{tikzcd}$$

$$Sh_{c,S}(\mathbb{R}) = \mathrm{Rep}\left(\begin{tikzcd}[row sep=small] & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd} \right)$$



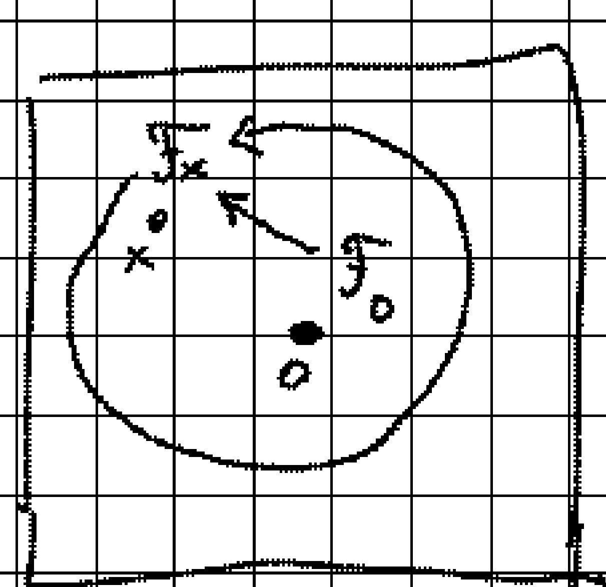
E.g. $j : \mathbb{R}_{>0} \hookrightarrow \mathbb{R}$

what is $j_* \tilde{k}$? $\qquad 0 \xleftarrow{0} k \xrightarrow{\sim} k$

$j_! \tilde{k}$? $\qquad 0 \xleftarrow{0} 0 \xrightarrow{0} k$

E.g. $X = \mathbb{C}$ $\quad S = \{0, \mathbb{C}^*\}$

$\mathcal{F} \in Sh_{c,S}$:



$F_0 \longrightarrow F_x \circlearrowleft m \qquad m = $ monodromy

$F_0 \longrightarrow F_x^m := \ker(m - 1) \qquad m$-invariants

This is because $EP(\mathbb{C}, S) = \left\{ \bullet \xrightarrow{a} \bullet \circlearrowleft m, m^{-1} \;\middle|\; \begin{matrix} ma = a \\ m m^{-1} = 1 \end{matrix} \right\}$

$$Sh_{c,S}(\mathbb{C}) = \mathrm{Rep}\left[\bullet \xrightarrow{a} \bullet \circlearrowleft m, m^{-1} : ma = a \right]$$

Alternate quiver

$$\begin{tikzcd} \tilde{0} \arrow[r, bend left, "p"] & x \arrow[l, bend left, "q"] \end{tikzcd}$$

related to previous quiver via
$\tilde{0} = \mathrm{Cone}(a) \qquad x = x.$

$1 - qp$ inv.

$1 - pq$ inv.

5 favourite objects

1) k_0
2) $k_{\mathbb{C}}$ these are all the indecomposables
3) $j_* k_{\mathbb{C}^*}$
4) $j_! k_{\mathbb{C}^*}$
5) ?

In our two quivers, these look like:

$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[loop right, "m"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, bend left, "p"] & \bullet \arrow[l, bend left, "q"] \end{tikzcd}$$

1) k_0 $\quad k \longrightarrow 0 \circlearrowleft \qquad \begin{tikzcd} k \arrow[r, bend left] & 0 \arrow[l, bend left] \end{tikzcd}$

2) $k_{\mathbb{C}}$ $\quad k \xrightarrow{\sim} k \circlearrowleft \qquad \begin{tikzcd} 0 \arrow[r, bend left] & k \arrow[l, bend left] \end{tikzcd}$

3) $j_* k_{\mathbb{C}^*}$ $\quad \begin{array}{lll} 1 & k & \\ 0 & k & \xrightarrow{\sim} k \end{array} \qquad \begin{tikzcd} k \arrow[r, bend left, "\sim"] & k \arrow[l, bend left, "0"] \end{tikzcd}$

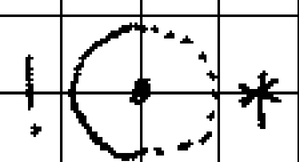
(NB: Rj_*)

4) $j_! k_{\mathbb{C}^*}$ $\quad 0 \longrightarrow k \circlearrowleft ? \qquad \begin{tikzcd} k \arrow[r, bend left, "0"] & k \arrow[l, bend left, "\sim"] \end{tikzcd}$

5) ? $\quad k[-1] \qquad k \qquad \begin{tikzcd} k \oplus k \arrow[r, bend left, "\pi_1"] & k \arrow[l, bend left, "\iota_2"] \end{tikzcd}$

tilting sheaf.

Sheaf ? looks like this:



$j_{!/*} k_{\mathbb{C}^*}$ - sections can approach 0 from one side but not the other.

E.g. $X = S^1$ $\quad \mathcal{S} = \{0, S^1 \setminus 0\}$

$\mathcal{F}_0 \rightleftarrows \mathcal{F}_*$.

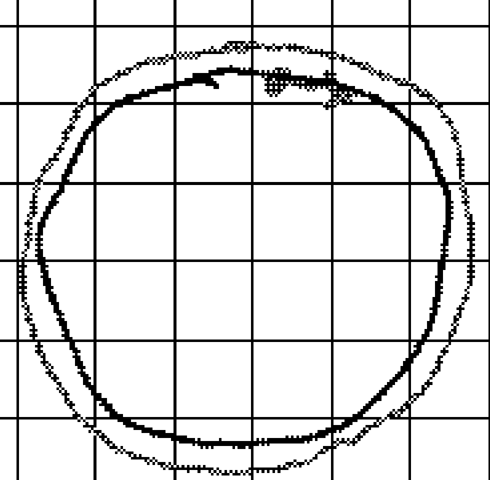
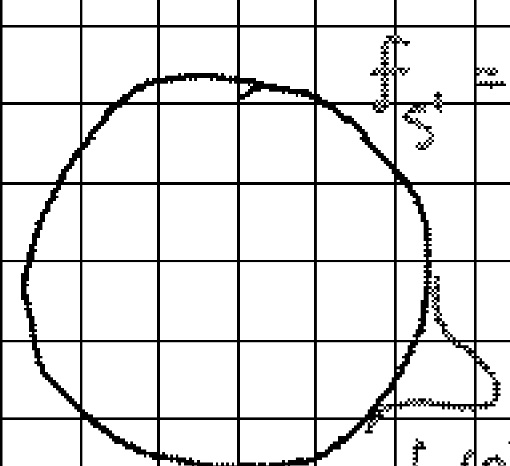
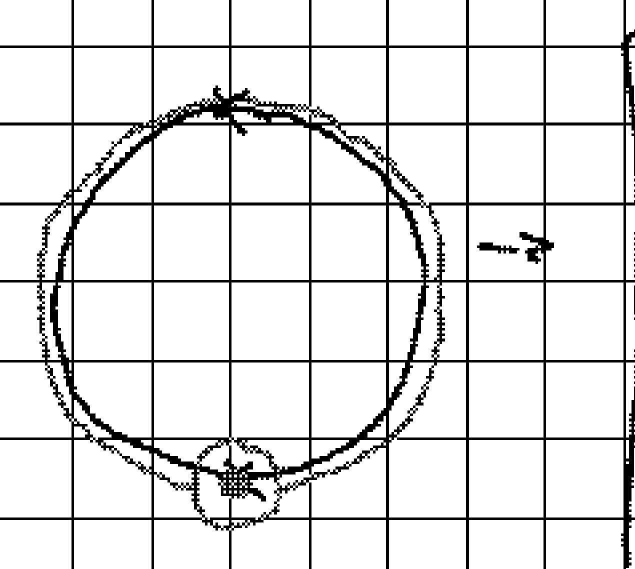
Nadler- overview

Note Title 6/13/2011

Unity of quantum geometry of symplectic manifolds

Warmup: cohomology of mfd X (E.g. $X = S^1$, $H^*(X) = \begin{matrix} 0 & k \\ 1 & k \end{matrix}$ ← coeff. field)

Three viewpoints:

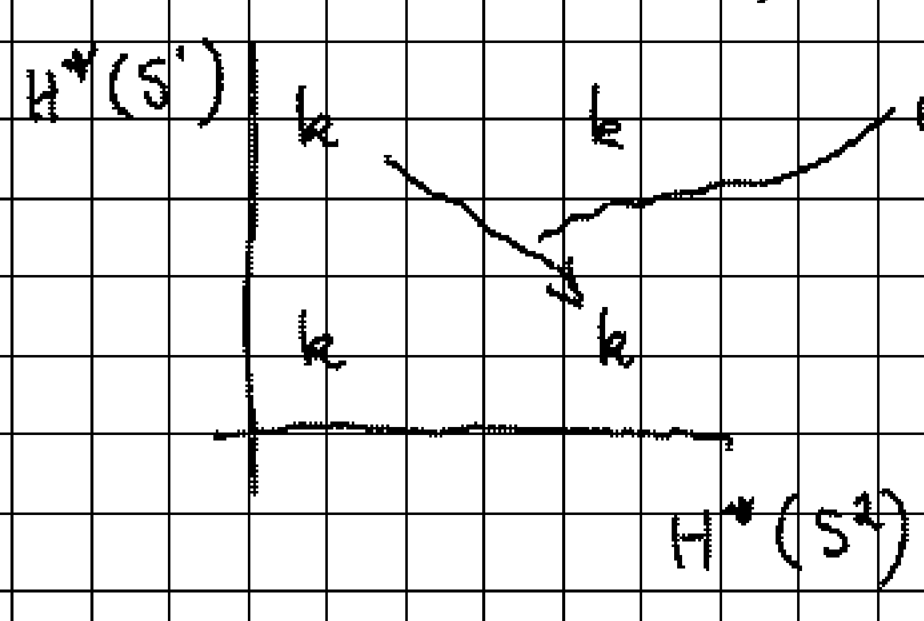
Topological	Algebraic	Analytic
$C^*(X)$ = sing. cochains	$\Omega^*(X)$ = diff. forms	$M(X)$ = Morse complex
	 f_{S^1} = const $b(\theta)d\theta$	

Rmks: 1) "quantum" is already here: linearity, we can 'add' two points, like superposition.

2) "cohomology" should mean the chain complex.

E.g. $S^3 \xrightarrow{\text{Hopf}} S^2$

Spectral sequence



extra information: comes from keeping cochain-level info about $C^*(S^2)$, $C^*(S^1)$.

Symplectic manifolds

M, ω ← closed nondeg. 2-form

Darboux: locally $\mathbb{R}^{2n}$, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

"Quantum": noncommutative deformation

functions x_i, y_j, $x_i \cdot y_j - y_j \cdot x_i = \delta_{ij}$

What to study: submanifolds and other geom. objects. Quantum means they "make sense" after such a non-com. deformation.

E.g. If we want to talk about submanifolds, we want to be able to locally define our submanifold by a zero locus of functions, and the functions must commute (if x_i and y_i vanish on $N \subset M$, then so does $x_i y_i - y_i x_i = 1$).

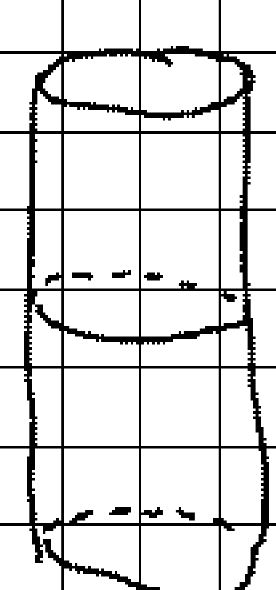
$\rightsquigarrow$ Uncertainty Principle: $N \subset M$ must be coisotropic
in particular smallest submfds are $L \subset M$ Lagrangians (no longer points).

So Lagrangians are the smallest building blocks.

We want to associate, to (M, ω), a category whose objects are Lagrangians (or more generally coisotropic submanifolds), and whose morphisms are quantum interactions.

Basic case: $M = T^*X$.

E.g. $X = S^1$



Top.	Alg.	Anal.
Constructible sheaves on X	(X complex) D-modules on X	Infinitesimal Fukaya category
"Lagrangian theory"	"coisotropic theory"	"Lagrangian theory"
$\cup$	$\uparrow$	$\cup$
Const. sheaves w.r.t. a fixed stratification	richer theory: contains the others as subcategory.	wrapped Fuk. cat. w.r.t. fixed Lag. skeleton
	$\cup$	
	... constructible D-modules w.r.t. a fixed stratification	

Why do we have a nice picture here?

1) We have a contracting dilation (exact str.)

2) Polarization: Lagrangian foliation by fibres of $T^*X \longrightarrow X$

E.g. 1) M Kähler $\subset \mathbb{CP}^n$

$\cup \qquad \cup$

$H = \mathbb{CP}^{n-1}$

$M \setminus (M \cap H)$ $\uparrow$ exact

2) S = Kleinian surface singularity

$\uparrow$

$\tilde{S}$ symplectic resolution

Looks like "a cotangent bundle of $\mathbb{P}^1$ but with a zero section which is a tree of $\mathbb{P}^1$'s." (cf. Day 4)

General principle: Quantum geometry of (M, ω) with polarization F should be equivalent to classical geometry of M/F.

<u>Day 4 polarised examples:</u>

1) $$\begin{tikzcd} M = T^*(S^1)^n \arrow[r] \arrow[d] & (S^1)^n \\ (\mathbb{R}^n)^\vee & \end{tikzcd}$$

two interesting foliations $\rightarrow$ relate

1) A_∞ structures

2) Cones, triangles, envelopes

3) Homological Perturbation

Motivation:

$$\begin{tikzcd}
 & \Omega^\bullet_{deR}(X) & & \\
Tw\,Open(X) \arrow[r, leftrightarrow, "(2)"] \arrow[dd, "\simeq"', "(2)"] & Open(X) \arrow[u, leftrightarrow] \arrow[r, "(3)"'] \arrow[ddl] & Mor(X) \arrow[d, "(2)"] & \\
 & & Tw\,Mor(X) \arrow[dl, "\cong"] & \\
Sh(X) \arrow[rr, "{N, NZ}"'] & & Fuk(T^*X)\ (1) &
\end{tikzcd}$$

(1) A_∞ structures.

Example: X a topological space, $x \in X$

$$\Omega X = \{ \gamma : S^1 \to X \mid \gamma(1) = x \}$$

$$\pi_0 \Omega X = \pi_1(X)$$

$$\Omega X \times \Omega X \longrightarrow \Omega X$$ product - only defined up to reparametrisation.

For each $n \geq 2$ we have a space of possible compositions

$$\mathcal{O}(n) \times \underbrace{\Omega X \times \dots \times \Omega X}_{n} \longrightarrow \Omega X$$

where $\mathcal{O}(n) = \mathrm{Emb}(\coprod_n I, I)$

A space w/ such a structure is an A_∞ algebra. (in spaces)

Example of A_∞ categories

X a space

$\Pi_\infty X$ has objects = points of X.

$\mathcal{M}or(x_0, x_1) = \{\gamma : I \to X : \gamma(0) = x_0, \gamma(1) = x_1\}$

<u>Defn</u>: (A_∞ algebra in chain complexes)

Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ be a graded vector space

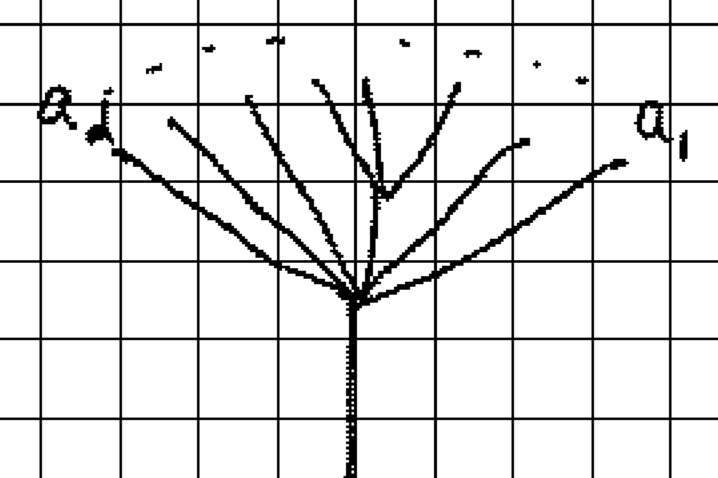
Suppose we have maps ($\forall\, d \geq 1$)

$$\mu^d : V^{\otimes d} \longrightarrow V[2-d]$$

satisfying

$$\sum_{i,k} (-1)^{*(i)} \mu^{d-k+1}\left(a_d, \ldots, a_{i+k+1}, \mu^k(a_{i+k}, \ldots, a_{i+1}), a_i, \ldots, a_1\right) = 0$$

or, graphically,

$\sum$  $= 0$.

The sign convention is

$$*(i) = \sum_{j=1}^{i} |a_j| - i.$$

<u>Obs</u>: $d = 1$: μ^1 has deg 1 and $\mu^1 \circ \mu^1 = 0$

$d = 2$: $m(a_2, a_1) = \mu^2(a_2, a_1) \cdot (-1)^{|a_1|}$

μ^1 is a derivation for m.

(i.e. satisfies the Leibniz rule)

$d = 3$: $\delta \mu^3 + \mu^3 \delta$ = associator of m.

$\Rightarrow$ the homology $H(A) := H^*(A, \mu^1)$ is an associative graded algebra.

<u>Defn</u>: A_∞ category $\mathcal{A}$ has

- objects $\mathrm{Ob}\,\mathcal{A}$
- hom spaces $\mathcal{A}(x_0, x_1)$ are $\mathbb{Z}$-graded vector spaces
- $\mu^d : \mathcal{A}(x_{d-1}, x_d) \otimes \ldots \otimes \mathcal{A}(x_0, x_1) \longrightarrow \mathcal{A}(x_0, x_d)[2-d]$.

  composition maps.

Defn: Let $f: A \to B$ be a morphism of A_∞ algebras. If $H(f): H(A) \to H(B)$ is an isomorphism, we say f is a quasi-isomorphism.

Similarly, if $f: \mathcal{A} \to \mathcal{B}$ is a morphism (functor) of A_∞ categories such that $H(f): H(\mathcal{A}) \to H(\mathcal{B})$ is an equivalence of graded linear categories, we say f is a quasi-equivalence.

If $\mu^d = 0$ for all $d \geq 3$, we use the term "dg" (differential graded) rather than "A_∞".

E.g. $\Omega^{\cdot}_{deR}(X)$, for X a C^∞ mfd, with de Rham differential and wedge product.

(2) The category of chain complexes (Ch_k is a dg-category (k = base field)).

Some properties:

(1) Ch_k has a zero object (not unique - any acyclic complex is a zero object)

(2) Ch_k has mapping cones / cokernels

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[ddr, bend right] & Y \arrow[d] \arrow[dd, bend left] \\
0 \arrow[r] & C(f) \arrow[d, dotted] \\
 & Z
\end{tikzcd}$$

Namely, $C(f) := X[1] \oplus Y$

$$d_{C(f)} = \begin{bmatrix} d_X & 0 \\ f & -d_Y \end{bmatrix}$$

(3) Ch_k has a shift functor $T: Ch_k \longrightarrow Ch_k$
$$X \longmapsto X[1]$$

As a consequence, $H^0 Ch_k$ is <u>triangulated</u>.

<u>The Yoneda Embedding</u>:

Given an A_∞ category $\mathcal{A}$, we can look at

$$A_\infty \text{ fun } (\mathcal{A}^{op}, Ch_k) =: A_\infty\text{-modules over } \mathcal{A}.$$

The Yoneda embedding

$$\mathcal{A} \hookrightarrow \mathcal{A}\text{-mod}$$
$$X \longmapsto \mathcal{A}(\cdot, X)$$

This is cohomologically full and faithful (when $\mathcal{A}$ is a c-unital category)

<u>Cor</u>: Every A_∞ cat. is quasi-equiv. to a dg. cat.

<u>Rmk</u>: The triangulated envelope of $\mathcal{A}$ is the smallest subcategory of $\mathcal{A}$-mod generated generated by cones and shifts of the image of $\mathcal{A}$.

<u>Rmk</u>: "Tw $\mathcal{A}$" is a model for the Δ envelope of A.

As a consequence, $H^0 Ch_k$ is triangulated

$$\text{Perf } A := \Delta \text{ envelope of } A$$
$$\cap$$
$$A\text{-mod}.$$

(3) Let $\mathcal{B}$ be an A_∞ cat. and $\forall$ objects $X_0, X_1 \in \mathrm{Ob}\,\mathcal{B}$,

$$G : \mathcal{B}(X_0, X_1) \to \mathcal{B}(X_0, X_1) \quad \text{s.t. } G^2 = G.$$

and G is a chain map.

Let F be the map.

$$F : G(\mathcal{B}(X_0, X_1)) \hookrightarrow \mathcal{B}(X_0, X_1).$$

Suppose $\exists$ $T : \mathcal{B}(X_0, X_1) \circlearrowleft$, ~~s.t.~~ deg -1 s.t.

$$F \circ G - id = \mu^1_{\mathcal{B}} T + T \mu^1_{\mathcal{B}}$$

Then $\exists$ an A_∞ cat $\mathcal{A}$ w/ $\mathrm{Ob}\,\mathcal{A} = \mathrm{Ob}\,\mathcal{B}$

$$\mathcal{A}(X_0, X_1) = G(\mathcal{B}(X_0, X_1))$$

s.t. F, G extends to a quasi-equivalence

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}.$$

E.g. $\mathcal{B}$ dg cat w/ one object. X a C^∞ mfd.

$$\begin{array}{l} \mathcal{B}(\cdot,\cdot) = C^\cdot_X \quad \text{"singular cells"} \\ \quad \downarrow G \\ M^\cdot_X \quad \text{Morse cells} \end{array}$$

$G(\alpha) =$ Morse flow applied to α, flowing it onto some Morse cells. $\lim_{t\to\infty} \Phi_t(\alpha)$

$$T = \bigcup_{t \geq 0} \Phi_t(\alpha)$$

Ian Shipman

Note Title 6/13/2011

$f_1, \dots, f_r$ real analytic on $\mathbb{R}^n$

$\{f_i > 0\}, \{f_i = 0\}, \{f_i < 0\}$

Partition of $\mathbb{R}^n$ given by intersections

f_i are linear $\Rightarrow$ partition by polyhedra.

Consider finite unions to get a collection that is closed under
- finite intersections
- finite unions
- complement

Also closed under f^{-1} where $f: \mathbb{R}^n \to \mathbb{R}^n$ $\mathbb{R}$-analytic.

Extend to $\mathbb{R}$-analytic manifolds

Defn: A semi-analytic set $V \subseteq M$ (M $\mathbb{R}$-analytic mfd) is one which is locally represented by inequalities of analytic functions.

Prop: Let $U \subset M$ be open, $f_1, \dots, f_r$ be analytic on U, and $p \in U$. Then:

$\exists$ $f_{r+1}, \dots, f_{r+k}$ analytic in a nbhd of p, s.t.

1) Each part in the partition assoc. to $f_1, \dots, f_{r+k}$

2) If $A = \bigcap_{i=1}^{r+k} \{f_i \gtreqless 0\}$

then $\overline{A} = \bigcap_{i=1}^{r+k} \{f_i \geq / = / \leq 0\}$

$\Rightarrow$ closure, interior, components of semianalytic sets are semianalytic. Moreover, component set is locally finite.

E.g. Images are <u>not</u> always semianalytic.

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\phi(x_1, x_2) = (x_1, x_1x_2, x_1x_2 e^{x_2})$$

$\phi(D^2)$ is <u>not</u> semi-analytic:
for any f analytic near $0 \in \mathbb{R}^3$, $f \circ \phi = 0$
$\Rightarrow f = 0$.

<u>Defn</u>: M is $\mathbb{R}$-analytic. $V \subseteq M$ is <u>subanalytic</u> if locally on M there is a partition $V = \coprod_i V_i$ where V_i is either semi-analytic or of the form $\phi(A)$ where

- $\underset{\text{analytic}}{A} \subset \underset{\mathbb{R}\text{-analytic mfd}}{N}$
- $\phi : N \to M$ analytic
- $\phi|_A$ is proper.

<u>Subanalytic sets form a Boolean algebra, i.e. closed under</u>

- finite union
- finite intersection
- complement
- inverse image (analytic)
- proper image (analytic)
- closure, interior, components

also, component set is locally finite.

Mappings:

$$\begin{array}{ccc} M & & N \\ \cup\!| & & \cup\!| \\ A & \xrightarrow[\phi]{} & B \end{array}$$

sub or semi-analytic subsets

Defn: ϕ is sub or semianalytic if $\Gamma_\phi \subset M \times N$ is sub or semi analytic.

NB: $\not\Rightarrow$ continuous

Inverse image and proper image under sub- and semi-analytic mapping are the same type.

Ex: $V \subset \mathbb{R}^n$ subanalytic

$g : \mathbb{R}^n \to \mathbb{R}$

$g(y) = d(y, \overline{V})$

is subanalytic. So if $V = \overline{V}$ then $V = g^{-1}(0)$ and V is defined by g.

$\pi : \mathbb{R}^{n+m} \to \mathbb{R}^m$

$V \subset \mathbb{R}^{n+m}$ rel. compact

$\pi(V)$ is subanalytic

$\exists\ B_i \subseteq \mathbb{R}^{n+m}$ finite # of smooth semianalytic

$\bigcup \pi(B_i) = \pi(V)$

$\pi|_{B_i} : B_i \longrightarrow \mathbb{R}^m$ is immersive

$\exists$ common complement to $\pi_* T_x B_i \quad x \in B_i$.

$\Rightarrow$ Thus subanalytic sets are global images of $\mathbb{R}$-analytic mflds. (Uniformisation)

Defn (condition B): $X, Y \subset \mathbb{R}^n$ C^1-submanifolds

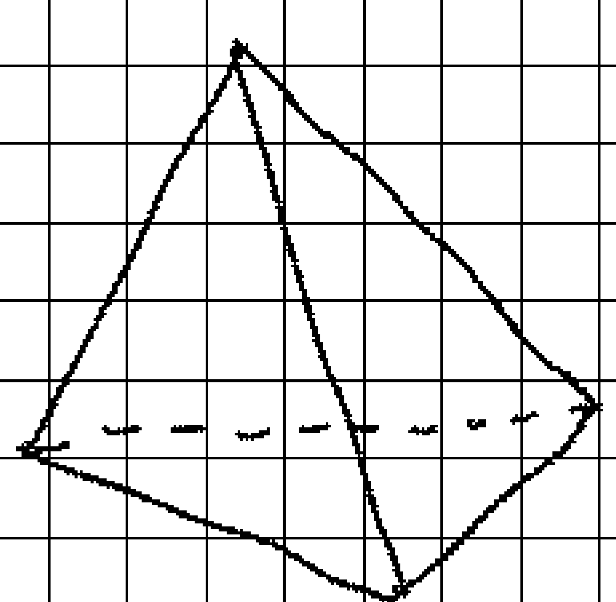
Pair (X, Y) satisfies condition B if for any $y \in Y$ whenever

$$x_i \overset{\in X}{\longrightarrow} y \qquad y_i \overset{\in Y}{\longrightarrow} y$$

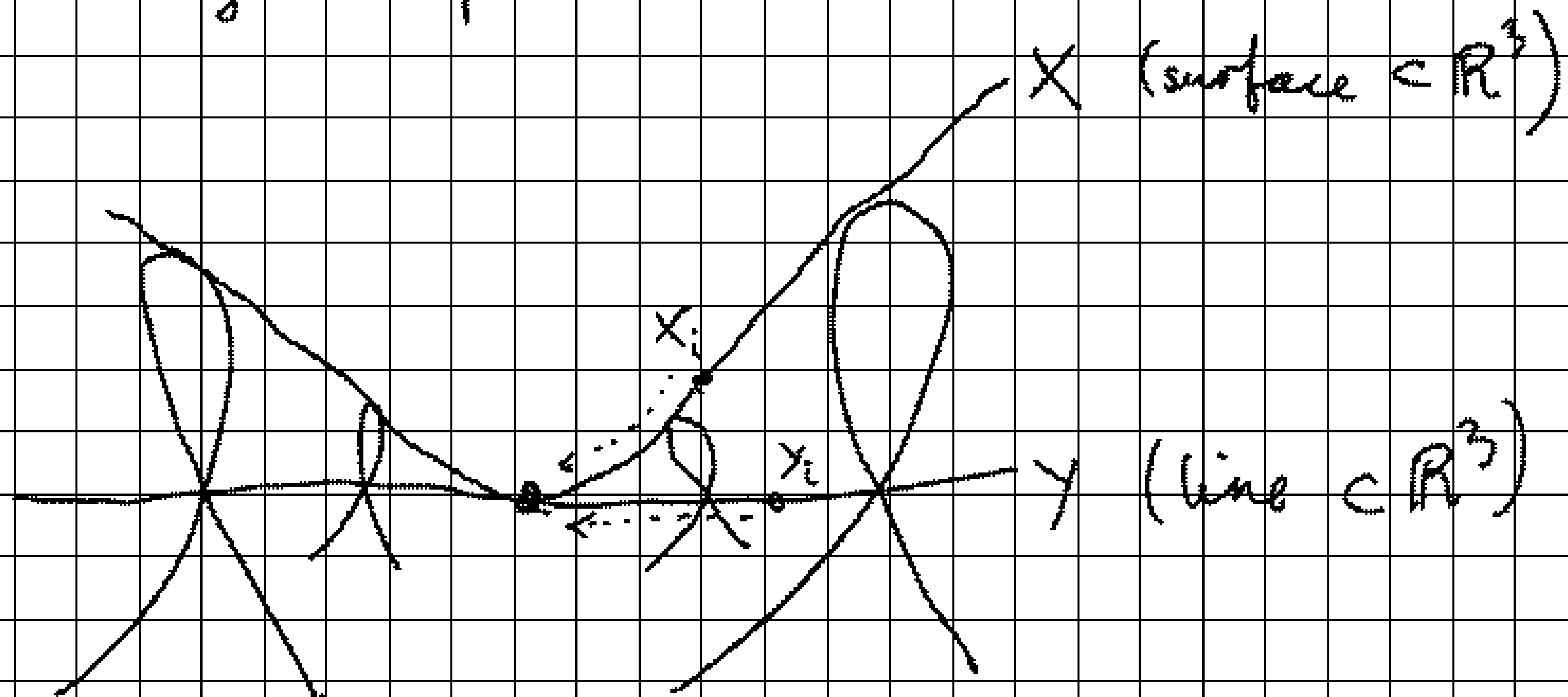
$$\lim T_{x_i} X = \tau, \quad \lim \overline{x_i y_i} = \ell$$

then $\ell \subseteq \tau$.

E.g.



E.g. Whitney's cusp



does not satisfy condition B.

condition B says roughly "the intersection of X, Y looks locally the same along Y".

In Whitney example, middle point: other points

NB. There's a condition A, implied by B, which says the union of conormals of strata is closed.

Defn (Whitney stratification):

$V \subseteq M$ locally compact (C').

$V = \bigcup_i V_i$ strata

1) $\{V_i\}$ locally finite

2) V_i is a locally closed submfd

3) (condition of the frontier)

If $V_i \cap \overline{V_j} \neq \phi$, $V_i \subseteq \overline{V_j}$

4) Every pair (V_i, V_j) satisfies condition B.

NB: Every subanalytic set has a Whitney stratification by analytic strata.

Defn: (stratified map)

A, B Whitney stratified

$\phi: A \to B$ is stratified if

1) $X \subseteq B$ is a stratum $\Rightarrow$ $\phi^{-1}(X)$ is a union of components of strata of A.

2) $Y \subseteq \phi^{-1}(X)$, $\phi|_Y : Y \to X$ is a submersion.
is a component of a stratum

Thom isotopy lemma:

If $\phi: V \to M$ is a stratified submersion then ϕ is a locally trivial fibration.

By a similar argument, if $\phi: V_1 \to V_2$ is stratified, and $f: V_2 \times I \to V_2$ stratified, then f lifts to a local isotopy on V_1, i.e.

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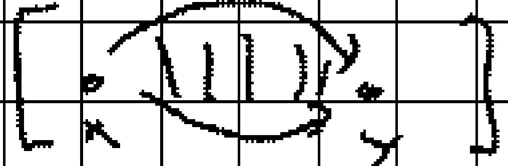
$\Big[$  $\Big]$ $\longleftrightarrow$ $\left[F_x \xrightarrow{\sim} F_y \right]$

homotopy
class of paths

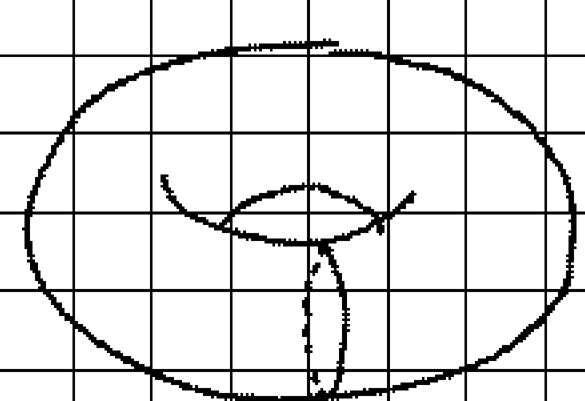
$$\mathrm{Loc}(X) \longleftrightarrow \mathrm{Rep}(\Pi_1 X) \longleftrightarrow \mathrm{Rep}(\pi_1(X))$$

fund. groupoid

Fix stratification $\mathcal{S}$, $X = \coprod X_\alpha$

$F \in Sh_{c,\mathcal{S}}(X) \Leftrightarrow F|_{X_\alpha}$ is a local system $\forall\ \alpha$

E.g.



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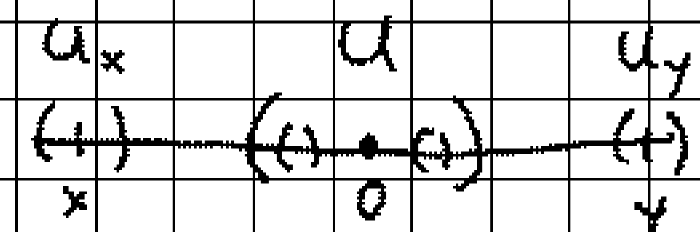
$$\left\{ \begin{matrix} \mathcal{S}\text{-constructible} \\ \text{sheaves on } X \end{matrix} \right\} \longleftrightarrow \left\{ \text{Representations of } EP(X, \mathcal{S}) \right\}$$

($\rightsquigarrow$ A_∞ functors, with higher homotopies of exit paths, if you talk about complexes of sheaves)

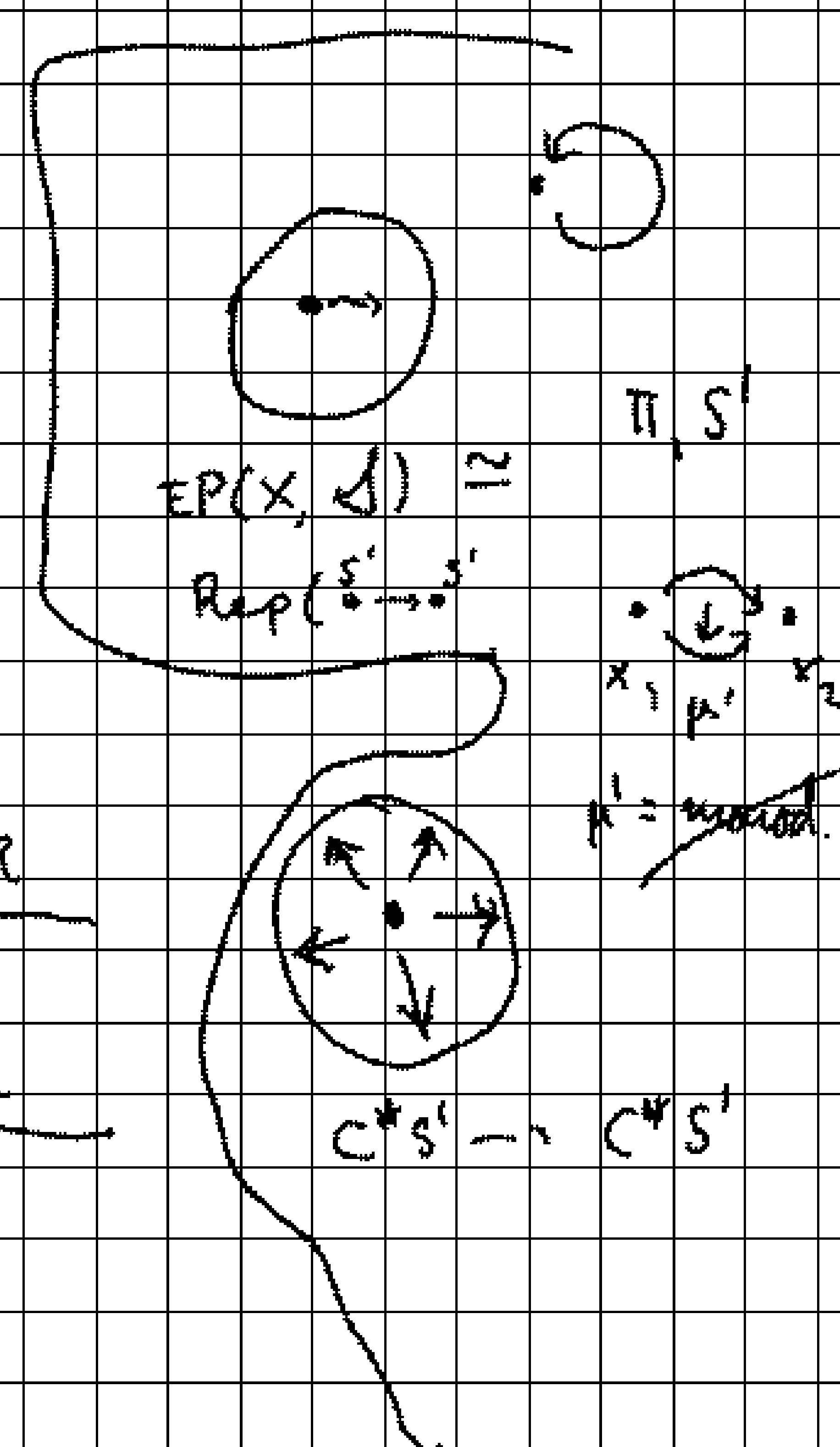
Eg: $X = \mathbb{R}$, $S = 0 \subset \mathbb{R}$

$F \in Sh_c(X)$



$$\begin{tikzcd} & F_0 = F(U) \arrow[dl] \arrow[dr] & \\ F(U_x) = F_x & & F_y = F(U_y) \end{tikzcd}$$

$Sh_{c,S}(\mathbb{R}) = \mathrm{Rep}(\bullet \leftarrow \bullet \rightarrow \bullet)$



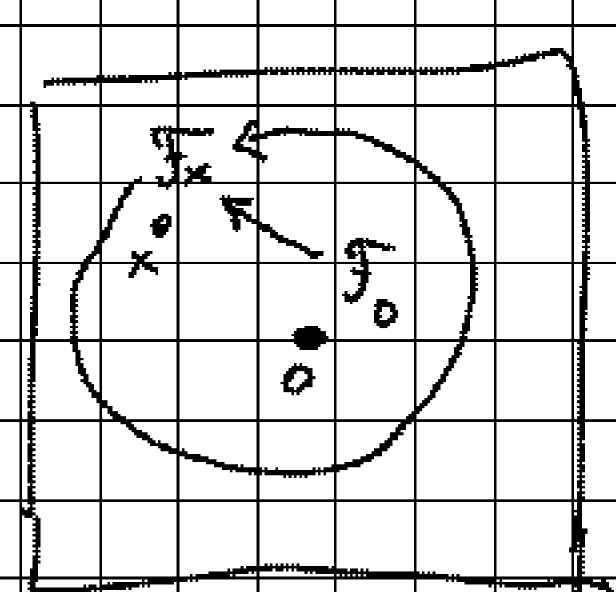
E.g. $j: \mathbb{R}_{>0} \hookrightarrow \mathbb{R}$

what is $j_* \tilde{k}$? $\qquad 0 \xleftarrow{0} k \xrightarrow{\sim} k$

$j_! \tilde{k}$? $\qquad 0 \xleftarrow{0} 0 \xrightarrow{0} k$

E.g. $X = \mathbb{C}$ $\quad \mathcal{S} = \{0, \mathbb{C}^*\}$

$\mathcal{F} \in Sh_{c,\mathcal{S}}$:



$F_0 \longrightarrow F_x \circlearrowleft m \qquad m = \text{monodromy}$

$F_0 \longrightarrow F_x^m := \ker(m-1) \qquad m\text{-invariants}$

This is because $EP(\mathbb{C}, \mathcal{S}) = \left\{ \bullet \xrightarrow{a} \bullet \circlearrowleft m, m^{-1} \;\middle|\; \begin{array}{l} ma = a \\ mm^{-1} = 1 \end{array} \right\}$

$Sh_{c,\mathcal{S}}(\mathbb{C}) = \mathrm{Rep}\left[\bullet \xrightarrow{a} \bullet \circlearrowleft m, m^{-1} : ma = a \right]$

Alternate quiver

$$\begin{tikzcd} \tilde{0} \arrow[r, bend left, "p"] & x \arrow[l, bend left, "q"] \end{tikzcd}$$

related to previous quiver via
$\tilde{0} = \mathrm{Cone}(a) \qquad x = x.$

$1 - qp$ inv.

$1 - pq$ inv.

5 favourite objects

1) k_0
2) $k_{\mathbb{C}}$ — these are all the indecomposables
3) $j_* k_{\mathbb{C}^*}$
4) $j_! k_{\mathbb{C}^*}$
5) ?

In our two quivers, these look like:

$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[loop right, "m"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, bend left, "p"] & \bullet \arrow[l, bend left, "q"] \end{tikzcd}$$

1) k_0 $\qquad \begin{tikzcd} k \arrow[r] & 0 \arrow[loop right] \end{tikzcd} \qquad\qquad \begin{tikzcd} k \arrow[r, bend left] & 0 \arrow[l, bend left] \end{tikzcd}$

2) $k_{\mathbb{C}}$ $\qquad \begin{tikzcd} k \arrow[r, "\sim"] & k \arrow[loop right] \end{tikzcd} \qquad\qquad \begin{tikzcd} 0 \arrow[r, bend left] & k \arrow[l, bend left] \end{tikzcd}$

3) $j_* k_{\mathbb{C}^*}$ $\qquad \begin{array}{lll} 1 & k & \\ 0 & k \xrightarrow{\sim} & k \end{array} \qquad\qquad \begin{tikzcd} k \arrow[r, bend left, "\sim"] & k \arrow[l, bend left, "0"] \end{tikzcd}$

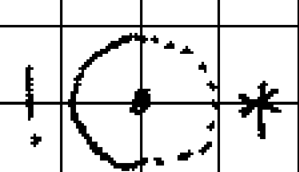
(NB: Rj_*)

4) $j_! k_{\mathbb{C}^*}$ $\qquad \begin{tikzcd} 0 \arrow[r] & k \arrow[loop right, "?"] \end{tikzcd} \qquad\qquad \begin{tikzcd} k \arrow[r, bend left, "0"] & k \arrow[l, bend left, "\sim"] \end{tikzcd}$

5) ? $\qquad k[-1] \qquad k \qquad\qquad \begin{tikzcd} k \oplus k \arrow[r, "\pi_1"] & k \arrow[l, bend left, "\iota_2"] \end{tikzcd}$

tilting sheaf.

Sheaf ? looks like this:



$j_{!/*} k_{\mathbb{C}^*}$ - sections can approach 0 from one side but not the other.

E.g. $X = S^1$ $\mathcal{S} = \{0, S^1 \setminus 0\}$

$\mathcal{F}_0 \rightleftarrows \mathcal{F}_*$.

Nadler - overview

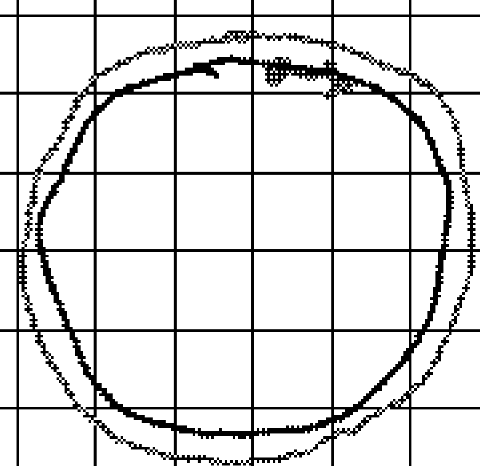
Note Title 6/13/2011

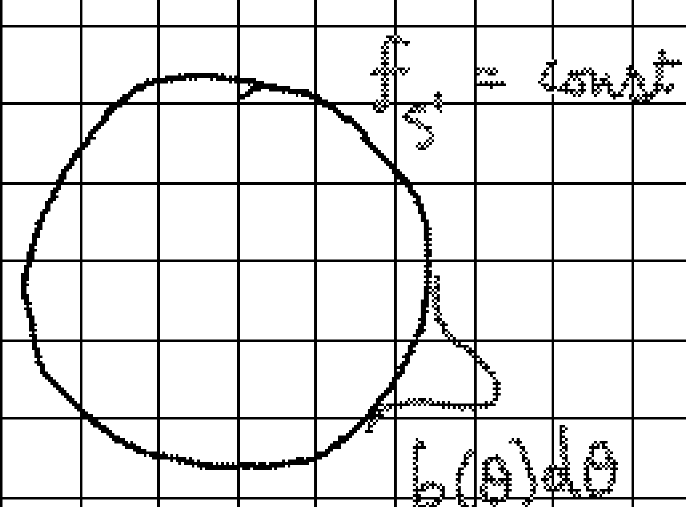
Unity of quantum geometry of symplectic manifolds

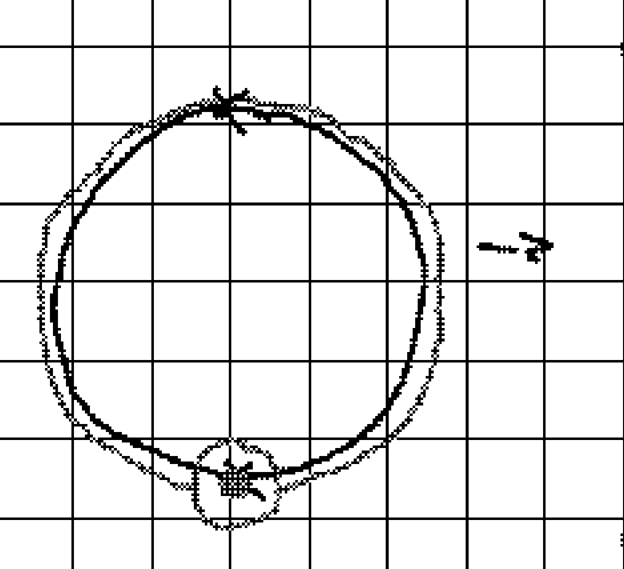
Warmup: cohomology of mfd X (E.g. $X = S^1$, $H^*(X) = \begin{matrix} 0 & k \\ 1 & k \end{matrix}$ ← coeff. field)

Three viewpoints:

Topological	Algebraic	Analytic
$C^*(X)$ = sing. cochains	$\Omega^*(X)$ = diff. forms	$M(X)$ = Morse complex







Rmks: 1) "quantum" is already here: linearity, we can 'add' two points, like superposition.

2) "cohomology" should mean the chain complex.

E.g. $S^3 \xrightarrow{\text{Hopf}} S^2$

Spectral sequence

$$\begin{array}{c|cc} H^*(S^1) & k & k \\ & k & k \\ \hline & & H^*(S^2) \end{array}$$

extra information: comes from keeping cochain-level info about $C^*(S^2)$, $C^*(S^1)$.

Symplectic manifolds

M, ω ← closed nondeg. 2-form

Darboux: locally $\mathbb{R}^{2n}$, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

"Quantum": noncommutative deformation

functions x_i, y_j, $\quad x_i \cdot y_j - y_j \cdot x_i = \delta_{ij}$

What to study: submanifolds and other geom. objects. Quantum means they "make sense" after such a non-com. deformation.

E.g. If we want to talk about submanifolds, we want to be able to locally define our submanifold by a zero locus of functions, and the functions must commute (if x_i and y_i vanish on $N \subset M$, then so does $x_i y_i - y_i x_i = 1$).

$\rightsquigarrow$ Uncertainty Principle: $N \subset M$ must be coisotropic

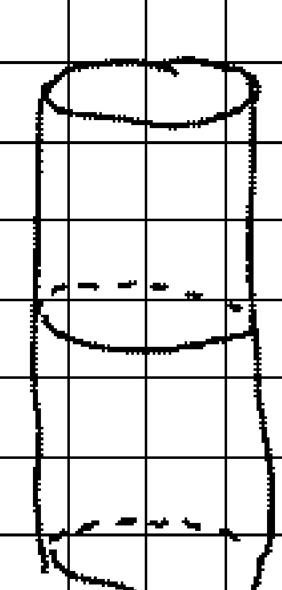
in particular smallest submflds are $L \subset M$ Lagrangians (no longer points).

So Lagrangians are the smallest building blocks.

We want to associate, to (M, ω), a category whose objects are Lagrangians (or more generally coisotropic submanifolds), and whose morphisms are quantum interactions.

Basic case: $M = T^*X$.

E.g. $X = S^1$



Top.	Alg.	Anal.
Constructible sheaves on X	(X complex) D-modules on X	Infinitesimal Fukaya category
"Lagrangian theory"	"coisotropic theory"	"Lagrangian theory"
$\cup$	$\uparrow$	$\cup$
Const. sheaves w.r.t. a fixed stratification	richer theory: contains the others as subcategory. $\cup$ ... constructible D-modules w.r.t. a fixed stratification	wrapped Fuk. cat. w.r.t. fixed Lag. skeleton

Why do we have a nice picture here?

1) We have a contracting dilation (exact str.)

2) Polarization: Lagrangian foliation by fibres of $T^*X \longrightarrow X$

E.g. 1) M Kähler $\subset \mathbb{CP}^n$

$$\begin{array}{ccc} M & & \mathbb{CP}^n \\ \cup & & \cup \\ M \setminus (M \cap H) & & H = \mathbb{CP}^{n-1} \end{array}$$

$\uparrow$ exact

2) S = Kleinian surface singularity

$\uparrow$

$\tilde{S}$ symplectic resolution

Looks like "a cotangent bundle of $\mathbb{P}^1$ but with a zero section which is a tree of $\mathbb{P}^1$'s." (cf. Day 4)

General principle: Quantum geometry of (M, ω) with polarization F should be equivalent to classical geometry of M/F.

Day 4 polarised examples:

1) $$\begin{array}{ccc} M = T^*(S^1)^n & \longrightarrow & (S^1)^n \\ \downarrow & & \\ (\mathbb{R}^n)^\vee & & \end{array}$$

two interesting foliations $\to$ relate

1) A_∞ structures

2) Cones, triangles, envelopes

3) Homological Perturbation

Motivation:

$$\begin{tikzcd}
 & & \Omega^\bullet_{deR}(X) \arrow[dl, leftrightarrow] & & \\
\mathrm{Tw}\,\mathrm{Open}(X) \arrow[d, "\cong"', "(2)"] & \mathrm{Open}(X) \arrow[l, leftrightarrow, "(2)"'] \arrow[rr, "(3)"'] \arrow[dl] & & \mathrm{Mor}(X) \arrow[dr, "(2)"] & \\
\mathrm{Sh}(X) \arrow[rrr, "{N, NZ}"'] & & & \mathrm{Fuk}(T^*X) \; (1) & \mathrm{Tw}\,\mathrm{Mor}(X) \arrow[l, "\cong"']
\end{tikzcd}$$

(1) A_∞ structures.

<u>Example</u>: X a topological space, $x \in X$

$$\Omega X = \{ \gamma : S^1 \to X \mid \gamma(1) = x \}$$

$$\pi_0 \Omega X = \pi_1(X)$$

$$\Omega X \times \Omega X \longrightarrow \Omega X$$ product - only defined up to reparametrisation.

For each $n \geq 2$ we have a space of possible compositions

$$\mathcal{O}(n) \times \underbrace{\Omega X \times \dots \times \Omega X}_{n} \longrightarrow \Omega X$$

where $\mathcal{O}(n) = \mathrm{Emb}(\coprod_n I, I)$

A space w/ such a structure is an A_∞ algebra. (in spaces)

<u>Example of A_∞ categories</u>

X a space

$\Pi_\infty X$ has objects = points of X.

$Mor(x_0, x_1) = \{\gamma : I \to X : \gamma(0) = x_0, \gamma(1) = x_1\}$

<u>Defn</u>: (A_∞ algebra in chain complexes)

Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ be a graded vector space

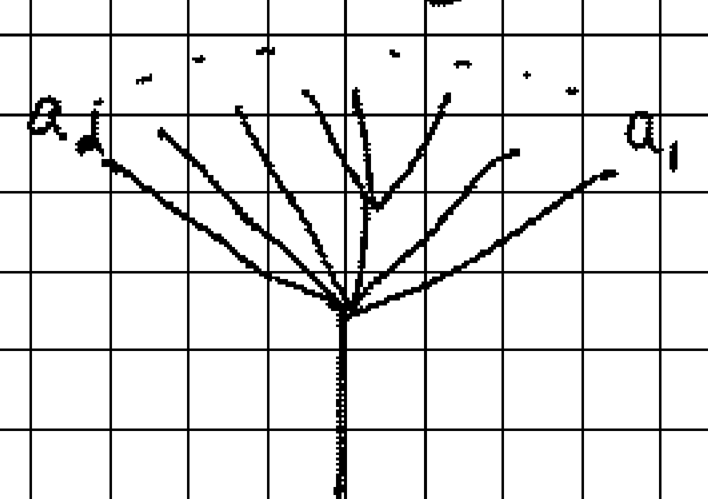
Suppose we have maps ($\forall\, d \geq 1$)

$$\mu^d : V^{\otimes d} \longrightarrow V[2-d]$$

satisfying

$$\sum_{i,k} (-1)^{*(i)} \mu^{d-k+1}\left(a_d, \dots, a_{i+k+1}, \mu^k(a_{i+k}, \dots, a_{i+1}), a_i, \dots, a_1\right) = 0$$

or, graphically,

$\sum$  $= 0$.

The sign convention is

$$*(i) = \sum_{j=1}^{i} |a_j| - i .$$

<u>Obs</u>: $d = 1$: μ^1 has deg 1 and $\mu^1 \circ \mu^1 = 0$

$d = 2$: $m(a_2, a_1) = \mu^2(a_2, a_1) \cdot (-1)^{|a_1|}$

μ^1 is a derivation for m.

(i.e. satisfies the Leibniz rule)

$d = 3$: $\delta \mu^3 + \mu^3 \delta$ = associator of m.

$\Rightarrow$ the homology $H^*(A) := H^*(A, \mu^1)$ is an associative graded algebra.

<u>Defn</u>: A_∞ category $\mathcal{A}$ has

- objects $Ob\, \mathcal{A}$
- hom spaces $\mathcal{A}(x_0, x_1)$ are $\mathbb{Z}$-graded vector spaces
- $\mu^d : \mathcal{A}(x_{d-1}, x_d) \otimes \dots \otimes \mathcal{A}(x_0, x_1) \longrightarrow \mathcal{A}(x_0, x_d)[2-d]$.

composition maps.

Defn: Let $f : A \to B$ be a morphism of A_∞ algebras. If $H(f) : H(A) \to H(B)$ is an isomorphism, we say f is a quasi-isomorphism.

Similarly, if $f : \mathcal{A} \to \mathcal{B}$ is a morphism (functor) of A_∞ categories such that $H(f) : H(\mathcal{A}) \to H(\mathcal{B})$ is an equivalence of graded linear categories, we say f is a quasi-equivalence.

If $\mu^d = 0$ for all $d \geq 3$, we use the term "dg" (differential graded) rather than "A_∞".

E.g. $\Omega^\cdot_{deR}(X)$, for X a C^∞ mfd, with de Rham differential and wedge product.

(2) The category of chain complexes (Ch_k is a dg-category (k = base field)).

Some properties:

(1) Ch_k has a zero object (not unique - any acyclic complex is a zero object)

(2) Ch_k has mapping cones / cokernels

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[ddr, bend right] & Y \arrow[d] \arrow[dd, bend left] \\
0 \arrow[r] & C(f) \arrow[d, dashed] \\
 & Z
\end{tikzcd}$$

Namely, $C(f) := X[1] \oplus Y$

$$d_{C(f)} = \begin{bmatrix} d_X & 0 \\ f & -d_Y \end{bmatrix}$$

(3) Ch_k has a shift functor $T: Ch_k \longrightarrow Ch_k$
$$X \longmapsto X[1]$$

As a consequence, $H^0 Ch_k$ is <u>triangulated</u>.

<u>The Yoneda Embedding</u>:

Given an A_∞ category $\mathcal{A}$, we can look at

$$A_\infty \text{fun}(\mathcal{A}^{op}, Ch_k) =: A_\infty\text{-modules over } \mathcal{A}.$$

The Yoneda embedding

$$\mathcal{A} \hookrightarrow \mathcal{A}\text{-mod}$$
$$X \longmapsto \mathcal{A}(\cdot, X)$$

This is cohomologically full and faithful (when $\mathcal{A}$ is a c-unital category)

<u>Cor</u>: Every A_∞ cat. is quasi-equiv. to a dg. cat.

<u>Rmk</u>: The triangulated envelope of $\mathcal{A}$ is the smallest subcategory of $\mathcal{A}$-mod generated generated by cones and shifts of the image of $\mathcal{A}$.

<u>Rmk</u>: "Tw $\mathcal{A}$" is a model for the Δ envelope of A.

As a consequence, $H^0 Ch_k$ is triangulated

$$\text{Perf } A := \Delta \text{ envelope of } A$$
$$\cap$$
$$A\text{-mod}.$$

(3) Let $\mathcal{B}$ be an A_∞ cat. and $\forall$ objects $X_0, X_1 \in \mathrm{Ob}\,\mathcal{B}$,

$$G : \mathcal{B}(X_0, X_1) \to \mathcal{B}(X_0, X_1) \quad \text{s.t. } G^2 = G.$$

and G is a chain map.

Let $\mathcal{F}$ be the map.

$$\mathcal{F} : G(\mathcal{B}(X_0, X_1)) \hookrightarrow \mathcal{B}(X_0, X_1).$$

Suppose $\exists\ T : \mathcal{B}(X_0, X_1) \circlearrowleft$, ~~s.t.~~ deg -1 s.t.

$$\mathcal{F} \circ \mathcal{G} - id = \mu^1_{\mathcal{B}} T + T \mu^1_{\mathcal{B}}$$

Then $\exists$ an A_∞ cat $\mathcal{A}$ w/ $\mathrm{Ob}\,\mathcal{A} = \mathrm{Ob}\,\mathcal{B}$

$$\mathcal{A}(X_0, X_1) = G(\mathcal{B}(X_0, X_1))$$

s.t. $\mathcal{F}, G$ extends to a quasi-equivalence

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B}.$$

E.g. $\mathcal{B}$ dg cat w/ one object. X a C^∞ mfd.

$$\begin{array}{l} \mathcal{B}(\cdot,\cdot) = C^\bullet_X \ \text{"singular cells"} \\ \quad\quad \Big\downarrow G \\ \quad\quad M^\bullet_X \ \text{Morse cells} \end{array}$$

$G(\alpha) =$ Morse flow applied to α, flowing it onto some Morse cells. $\lim_{t\to\infty} \Phi_t(\alpha)$

$$T = \bigcup_{t \geq 0} \Phi_t(\alpha)$$