Hiro - A_∞ structures

Note Title 6/13/2011

1) A_∞ structures

2) Cones, triangles, envelopes

3) Homological Perturbation

Motivation:

$$\begin{tikzcd}
 & & \Omega^{\cdot}_{deR}(X) \arrow[dl, leftrightarrow] & & \\
\mathrm{Tw\,Open}(X) \arrow[r, leftrightarrow, "(2)"] \arrow[d, "\simeq"', "(2)" near start] & \mathrm{Open}(X) \arrow[rr, "(3)"'] \arrow[dl] & & \mathrm{Mor}(X) \arrow[dr, "(2)"] & \\
\mathrm{Sh}(X) \arrow[rrr, "{N, NZ}"'] & & & \mathrm{Fuk}(T^*X) \; (1) & \mathrm{Tw\,Mor}(X) \arrow[l, "\cong"]
\end{tikzcd}$$

(1) A_∞ structures.

Example: X a topological space, $x \in X$

$$\Omega X = \{\gamma : S^1 \to X \mid \gamma(1) = x\}$$

$$\pi_0 \Omega X = \pi_1(X)$$

$$\Omega X \times \Omega X \longrightarrow \Omega X$$ product - only defined up to reparametrisation.

For each $n \geq 2$ we have a space of possible compositions

$$\mathcal{O}(n) \times \underbrace{\Omega X \times \dots \times \Omega X}_{n} \longrightarrow \Omega X$$

where $\mathcal{O}(n) = \mathrm{Emb}(\coprod_n I, I)$

A space w/ such a structure is an A_∞ algebra (in spaces).

Example of A_∞ categories

X a space

$\Pi_\infty X$ has objects = points of X.

$\mathcal{M}or(x_0, x_1) = \{\gamma : I \to X : \gamma(0) = x_0, \gamma(1) = x_1\}$

<u>Defn:</u> (A_∞ algebra in chain complexes)

Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ be a graded vector space

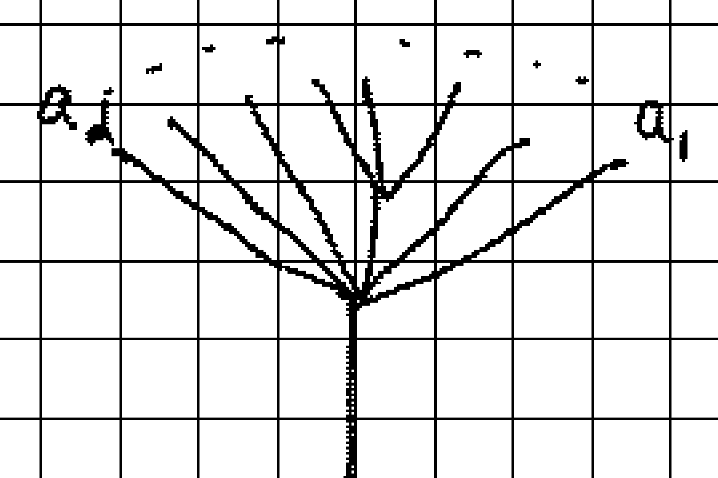
Suppose we have maps ($\forall\, d \geq 1$)

$$\mu^d : V^{\otimes d} \longrightarrow V[2-d]$$

satisfying

$$\sum_{i,k} (-1)^{*(i)} \mu^{d-k+1}\left(a_d, \dots, a_{i+k+1}, \mu^k(a_{i+k}, \dots, a_{i+1}), a_i, \dots, a_1\right) = 0$$

or, graphically,

$$\sum$$ 

$= 0$.

The sign convention is

$$*(i) = \sum_{j=1}^{i} |a_j| - i.$$

<u>Obs:</u> $d = 1$: μ^1 has deg 1 and $\mu^1 \circ \mu^1 = 0$

$d = 2$: $m(a_2, a_1) = \mu^2(a_2, a_1) \cdot (-1)^{|a_1|}$

μ^1 is a derivation for m.

(i.e. satisfies the Leibniz rule)

$d = 3$: $\delta \mu^3 + \mu^3 \delta$ = associator of m.

$\Rightarrow$ the homology $H(A) := H^*(A, \mu^1)$ is an associative graded algebra.

<u>Defn:</u> A_∞ category $\mathcal{A}$ has

- objects $\mathrm{Ob}\,\mathcal{A}$
- hom spaces $\mathcal{A}(x_0, x_1)$ are $\mathbb{Z}$-graded vector spaces
- $\mu^d : \mathcal{A}(x_{d-1}, x_d) \otimes \dots \otimes \mathcal{A}(x_0, x_1) \longrightarrow \mathcal{A}(x_0, x_d)[2-d]$.

composition maps.

Defn: Let $f : A \to B$ be a morphism of A_∞ algebras. If $H(f) : H(A) \to H(B)$ is an isomorphism, we say f is a quasi-isomorphism.

Similarly, if $f : \mathcal{A} \to \mathcal{B}$ is a morphism (functor) of A_∞ categories such that $H(f) : H(\mathcal{A}) \to H(\mathcal{B})$ is an equivalence of graded linear categories, we say f is a quasi-equivalence.

If $\mu^d = 0$ for all $d \geq 3$, we use the term "dg" (differential graded) rather than "A_∞".

E.g. $\Omega^{\cdot}_{deR}(X)$, for X a C^∞ mfd, with de Rham differential and wedge product.

(2) The category of chain complexes (Ch_k is a dg-category (k = base field)).

Some properties:

(1) Ch_k has a zero object (not unique - any acyclic complex is a zero object)

(2) Ch_k has mapping cones/cokernels

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[ddr, bend right] & Y \arrow[d] \arrow[dd, bend left] \\
0 \arrow[r] & C(f) \arrow[d, dotted] \\
& Z
\end{tikzcd}$$

Namely, $C(f) := X[1] \oplus Y$

$$d_{C(f)} = \begin{bmatrix} d_X & 0 \\ f & -d_Y \end{bmatrix}$$

(3) Ch_k has a shift functor $T : Ch_k \longrightarrow Ch_k$

$$X \longmapsto X[1]$$

As a consequence, $H^0 Ch_k$ is <u>triangulated</u>.

<u>The Yoneda Embedding</u>:

Given an A_∞ category $\mathcal{A}$, we can look at

$$A_\infty \text{ fun } (\mathcal{A}^{op}, Ch_k) =: A_\infty\text{-modules over } \mathcal{A}.$$

The Yoneda embedding

$$\mathcal{A} \hookrightarrow \mathcal{A}\text{-mod}$$
$$X \longmapsto \mathcal{A}(\cdot, X)$$

This is cohomologically full and faithful (when $\mathcal{A}$ is a c-unital category)

<u>Cor</u>: Every A_∞ cat. is quasi-equiv. to a dg. cat.

<u>Rmk</u>: The triangulated envelope of $\mathcal{A}$ is the smallest subcategory of $\mathcal{A}$-mod generated generated by cones and shifts of the image of $\mathcal{A}$.

<u>Rmk</u>: "$Tw\,\mathcal{A}$" is a model for the Δ envelope of A.

As a consequence, $H^0 Ch_k$ is triangulated

$$\begin{array}{l} \text{Perf } A := \Delta \text{ envelope of } A \\ \cap \\ A\text{-mod.} \end{array}$$

(3) Let $\mathcal{B}$ be an A_∞ cat. and $\forall$ objects $X_0, X_1 \in \mathrm{Ob}\,\mathcal{B}$,

$$G : \mathcal{B}(X_0, X_1) \to \mathcal{B}(X_0, X_1) \quad \text{s.t. } G^2 = G,$$

and G is a chain map.

Let $\mathcal{F}$ be the map.

$$\mathcal{F} : G(\mathcal{B}(X_0, X_1)) \hookrightarrow \mathcal{B}(X_0, X_1).$$

Suppose $\exists\ T : \mathcal{B}(X_0, X_1) \circlearrowleft$, ~~s.t.~~ deg -1 s.t.

$$\mathcal{F} \circ \mathcal{G} - id = \mu^1_{\mathcal{B}} T + T\mu^1_{\mathcal{B}}$$

Then $\exists$ an A_∞ cat $\mathcal{A}$ w/ $\mathrm{Ob}\,\mathcal{A} = \mathrm{Ob}\,\mathcal{B}$

$$\mathcal{A}(X_0, X_1) = G(\mathcal{B}(X_0, X_1))$$

s.t. $\mathcal{F}, G$ extends to a quasi-equivalence

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B}.$$

E.g. $\mathcal{B}$ dg cat w/ one object. $\quad X$ a C^∞ mfd.

$$\begin{array}{l} \mathcal{B}(\cdot,\cdot) = C^\cdot_X \quad \text{"singular cells"} \\ \qquad\qquad \downarrow G \\ \qquad\qquad M^\cdot_X \quad \text{Morse cells} \end{array}$$

$G(\alpha)$ = Morse flow applied to α, flowing it onto some Morse cells. $\lim_{t\to\infty} \Phi_t(\alpha)$

$$T = \bigcup_{t \geq 0} \Phi_t(\alpha)$$