Nadler - overview

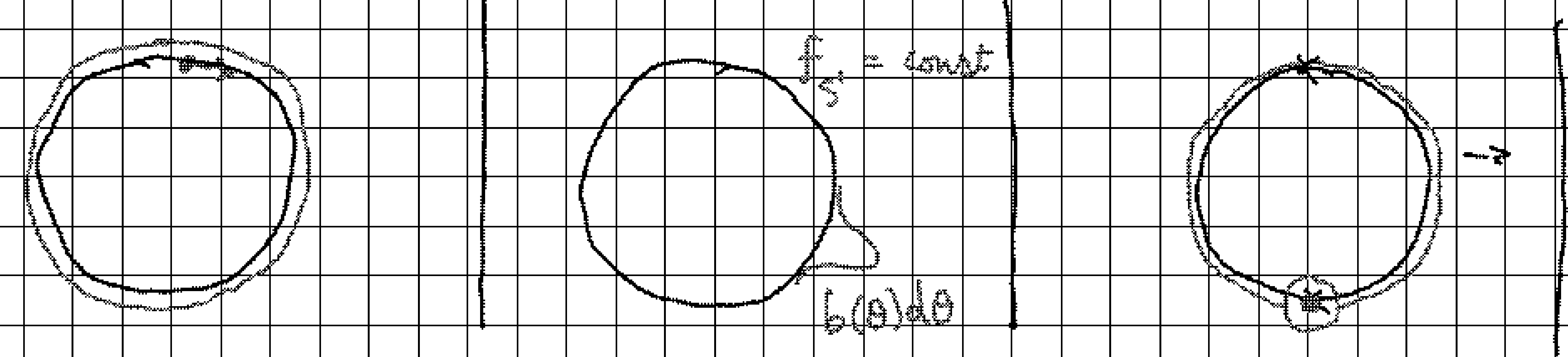
Note Title 6/13/2011

Unity of quantum geometry of symplectic manifolds

Warmup: cohomology of mfd X (E.g. $X = S^1$, $H^*(X) = \begin{matrix} 0 & k \\ 1 & k \end{matrix}$ ← coeff. field)

Three viewpoints:

Topological	Algebraic	Analytic
$C^*(X)$ = sing. cochains	$\Omega^*(X)$ = diff. forms	$M(X)$ = Morse complex

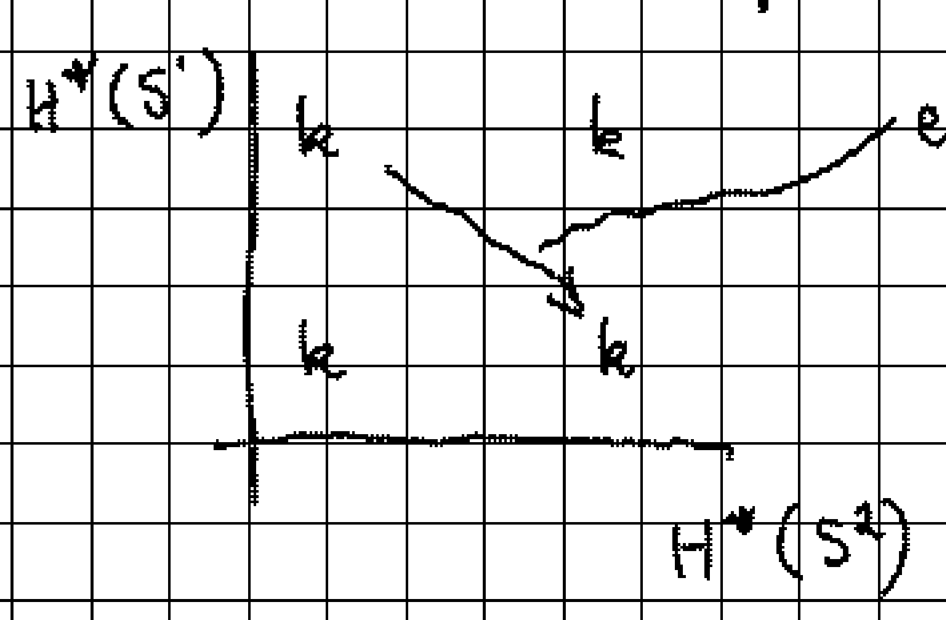


Rmks: 1) "quantum" is already here: linearity, we can 'add' two points, like superposition.

2) "cohomology" should mean the chain complex.

E.g. $S^3 \xrightarrow{\text{Hopf}} S^2$

Spectral sequence



extra information: comes from keeping cochain-level info about $C^*(S^2)$, $C^*(S^1)$.

Symplectic manifolds

M, ω ← closed nondeg. 2-form

Darboux: locally $\mathbb{R}^{2n}$, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

"Quantum": noncommutative deformation

functions x_i, y_j, $x_i \cdot y_j - y_j \cdot x_i = \delta_{ij}$

What to study: submanifolds and other geom. objects. Quantum means they "make sense" after such a non-com. deformation.

E.g. If we want to talk about submanifolds, we want to be able to locally define our submanifold by a zero locus of functions, and the functions must commute (if x_i and y_i vanish on $N \subset M$, then so does $x_i y_i - y_i x_i = 1$).

$\rightsquigarrow$ Uncertainty Principle: $N \subset M$ must be coisotropic

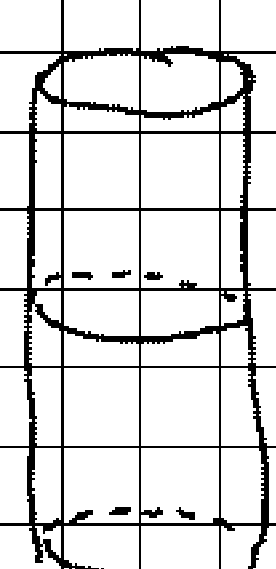
in particular smallest submflds are $L \subset M$ Lagrangians (no longer points).

So Lagrangians are the smallest building blocks.

We want to associate, to (M, ω), a category whose objects are Lagrangians (or more generally coisotropic submanifolds), and whose morphisms are quantum interactions.

Basic case: $M = T^*X$.

E.g. $X = S^1$



Top.	Alg.	Anal.
Constructible sheaves on X	(X complex) D-modules on X	Infinitesimal Fukaya category
"Lagrangian theory"	"coisotropic theory"	"Lagrangian theory"
$\cup$	$\uparrow$	$\cup$
Const. sheaves w.r.t. a fixed stratification	richer theory: contains the others as subcategory.	wrapped Fuk. cat. w.r.t. fixed Lag. skeleton
	$\cup$	
	... constructible D^*-modules w.r.t. a fixed stratification	

Why do we have a nice picture here?

1) We have a contracting dilation (exact str.)

2) Polarization: Lagrangian foliation by fibres of $T^*X \longrightarrow X$

E.g. 1)

$$\begin{array}{ccc} M & \text{Kähler} \subset & \mathbb{CP}^n \\ \cup & & \cup \\ & & H = \mathbb{CP}^{n-1} \\ M \setminus (M \cap H) & & \end{array}$$

$\uparrow$ exact

2) S = Kleinian surface singularity

$\uparrow$

$\tilde{S}$ symplectic resolution

Looks like "a cotangent bundle of $\mathbb{P}^1$ but with a zero section which is a tree of $\mathbb{P}^1$'s." (cf. Day 4)

General principle: Quantum geometry of (M, ω) with polarization F should be equivalent to classical geometry of M/F.

<u>Day 4 polarised examples:</u>

1) $$\begin{tikzcd} M = T^*(S^1)^n \arrow[r] \arrow[d] & (S^1)^n \\ (\mathbb{R}^n)^\vee & \end{tikzcd}$$

two interesting foliations $\to$ relate