Note Title 11/6/2013

Define the Floer cochain group,

$$CF^*(L_0, L_1) := \Lambda \langle L_0 \cap L_1 \rangle$$

Λ- vec. space with basis $= L_0 \cap L_1$,

where $\Lambda :=$ Novikov field

$$= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lambda_i \longrightarrow +\infty \right\}.$$

By putting some extra structure on the L_i, we can equip $CF^*(L_0, L_1)$ with a

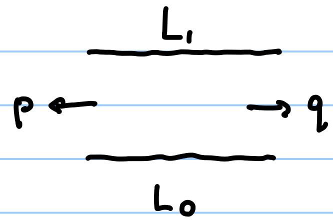
$$\mathbb{Z}/2N \text{ - grading},$$

$N =$ minimal Chern number of M.

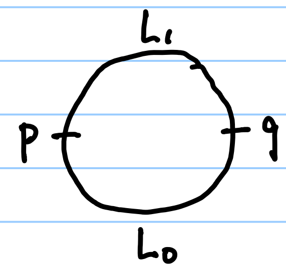
Let $J \in End(TM)$, $J^2 = -\mathbb{1}$ be an almost-complex structure compatible with ω.

Given $p, q \in L_0 \cap L_1$, define

$$\mathcal{M}(p;q) := \left\{ u : \mathbb{R} \times [0,1] \longrightarrow M \quad \begin{array}{l} Du \circ j = J \circ Du \quad J\text{-holomorphic} \\ u(s,0) \in L_0 \\ u(s,1) \in L_1 \\ \lim_{s \to +\infty} u(s,t) = p \\ \lim_{s \to -\infty} u(s,t) = q \end{array} \right\}$$



or



There's an $\mathbb{R}$-action on this moduli space,

$$c: \quad u(s,t) \longmapsto u(s+c,t).$$

Let $\mathcal{M}(p;q)_\beta :=$ maps in homotopy class $\beta \in \pi_2(M; L_0, L_1)$.

This is a <u>Fredholm problem</u>, so $\mathcal{M}(p;q)_\beta$ is generically a smooth manifold. Its dimension is given by the <u>Maslov index</u> $\mathrm{ind}(\beta)$ (comes from $\pi_1(\mathcal{L}Gr) \cong \mathbb{Z}$)

space of Lagrangian subspaces of a symplectic vector space.

We define

$$\partial: CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_1)$$

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ \mathrm{ind}(\beta)=1}} \#\left(\mathcal{M}(p;q)_\beta / \mathbb{R}\right) T^{\omega(\beta)} q$$

1-dim makes it 0-dim

Like for Gromov-Witten invariants, Gromov compactness $\Rightarrow$ $\#$ is finite for each β.

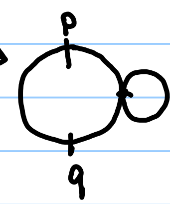
Thm (Floer): If $[\omega]\cdot\pi_2(M, L_i) = 0$, then

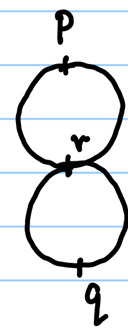
$$\partial^2 = 0 \mod 2.$$

Pf: Look at $\mathcal{M}(p;q)_\beta/\mathbb{R}$ for $\operatorname{ind}(\beta) = 2$.

Gromov compactness says, if we define $\overline{\mathcal{M}(p;q)_\beta/\mathbb{R}}$ by adding in nodal disks:

energy 'concentrates' near boundary point





energy 'concentrates' near $s = \pm\infty$.

energy 'concentrates' near interior point

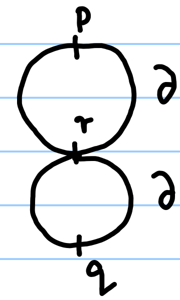
then $\overline{\mathcal{M}}(p;q)_\beta/\mathbb{R}$ is compact.

The hypothesis $[\omega]\cdot\pi_2(M,L)$ rules out everything on the left, because if u is J-holomorphic,

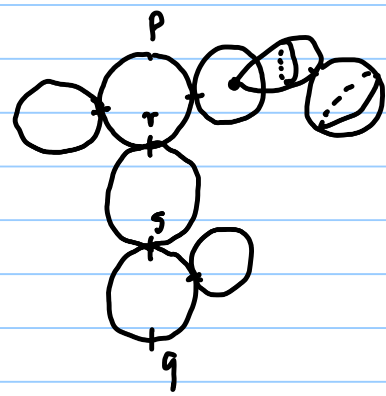
$$\begin{aligned} \omega(u) &= \int \omega(\partial_s u, \partial_t u)\, ds\, dt \\ &= \int \omega(\partial_s u, J\partial_s u)\, ds\, dt \\ &= \|\partial_s u\|_{L^2}^2 \end{aligned}$$

so $\omega(u) = 0 \Rightarrow u$ constant.

So we have a compact 1-manifold whose boundary points correspond to



(actually, the compactification includes all



but these configurations have index < 0, so generically they don't appear.)

the mod 2 count of boundary points in a compact 1-manifold is 0, so

$$\partial^2 = 0 \mod 2.$$

This allows us to define

$$HF^*(L_0, L_1) := H^*(CF^*(L_0, L_1), \partial)$$

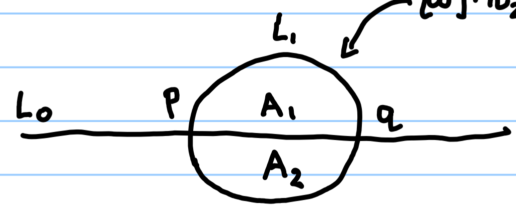
if we work over the coefficient field

$$\Lambda := \{ \sum a_i T^{\lambda_i} : a_i \in \mathbb{Z}_2 \}.$$

To work over Λ, we need to orient the moduli spaces, so elements are counted with sign, and the signed count of boundary points of a compact 1-manifold is 0.
It turns out you need your Lag.'s to be equipped with a Pin structure for this to work.

E.g. To show why we need to rule out disk bubbles:

$[\omega]\cdot\pi_2(\mathbb{C}, L_1) \neq 0.$

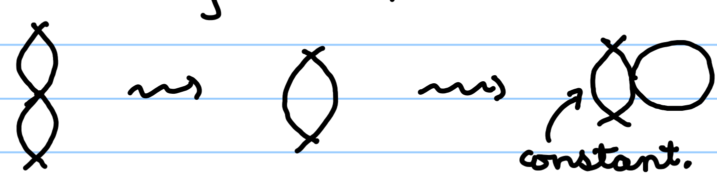


$$\partial p = T^{A_1} q$$

$$\partial q = T^{A_2} p$$

$\partial^2 p \neq 0$ because

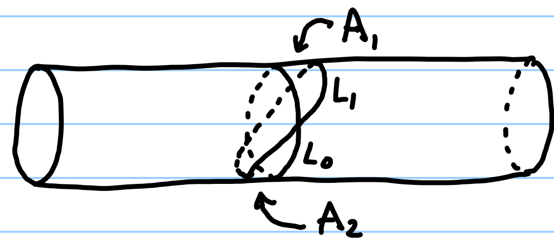
We get a disk bubbling off w/ boundary on L_1:



J-holomorphic disks with boundary on L_i obstruct the definition of Floer cohomology.

$HF^*(L_0, L_1)$ is independent of the choice of J used in its construction (as we will see soon).

E.g.



$$CF^*(L_0, L_1) \cong \Lambda\langle p, q\rangle$$

$\mathcal{M}(p;q)/\mathbb{R}$ = 2 points (2 strips from p to q)

$$\Rightarrow \partial(p) = T^{A_1} q - T^{A_2} q$$

$$\Rightarrow HF^*(L_0, L_1) \cong \begin{cases} \Lambda\langle p, q\rangle. & \text{if } A_1 = A_2 \\ 0 & \text{otherwise.} \end{cases}$$

This illustrates another important property of Floer cohomology: if

$$\varphi_H : M \longrightarrow M$$

is a Hamiltonian symplectomorphism, (i.e., the flow of a Ham. vector field X_H, where

$$\underset{\substack{\uparrow \\ \text{Ham. vector field}}}{\iota_{X_H}} \omega = \underset{\substack{\uparrow \\ \text{Ham. function}}}{dH}$$

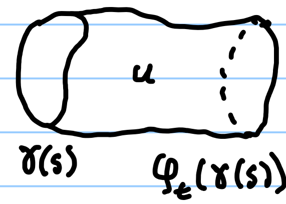
φ_H = time-1 flow of X_H), then

$$HF^*(\varphi_H(L_0), L_1) \cong HF^*(L_0, L_1).$$

(we say $\varphi_H(L)$ is <u>Hamiltonian isotopic</u> to L).

N.B. φ_t = time-t flow of Ham. vec. field X_H

$\Rightarrow$ for any loop,



$$\int u^* \omega = \int_0^1 \int_{S^1} \omega(X_H, \varphi_* \dot\gamma(s))\, ds\, dt$$

$$= \int_0^1 \int_{S^1} dH(\varphi_* \dot\gamma(s))\, ds\, dt$$

$$= \int_0^1 0\, dt = 0$$

$\Rightarrow$ area enclosed between L_1, $\varphi(L_1)$ is 0

$\Rightarrow$ $A_1 = A_2$.

So in fact, for L_0, L_1 loops around the cylinder,
$HF^*(L_0, L_1) \neq 0 \Leftrightarrow L_0, L_1$ Hamiltonian isotopic.

<u>Thm</u>: $HF^*(L_0, L_1)$ is independent of
- choice of J
- Hamiltonian isotopy of L_i.

<u>Pf</u>: First, recast our definition of HF^*: for Lag's L_0, L_1, function H,* a-c. structure J, Λ-generators of $CF^*(L_0, L_1; H, J)$ are time-1 flowlines of X_H from L_0 to L_1. These are in 1-1 correspondence with intersection points of $\varphi_H(L_0)$ and L_1, so

$$CF^*(L_0, L_1; H, J) \cong CF^*(\varphi(L_0), L_1; 0, J).$$

* such that $\varphi_H(L_0) \pitchfork L_1$

For generators $p, q : [0,1] \to M$, define $\mathcal{M}'(p,q)$ to be the set of solutions to

$$u : \mathbb{R} \times [0,1] \to M,$$

$$\begin{cases} \dfrac{\partial u}{\partial s} = J'\left(\dfrac{\partial u}{\partial t} - X_H\right) \\ u(s,0) \in L_0 \\ u(s,1) \in L_1 \\ \lim\limits_{s \to -\infty} u(s,t) = p(t) \\ \lim\limits_{s \to +\infty} u(s,t) = q(t) \end{cases}$$

and set

$$\partial : CF^*(L_0, L_1; H, J) \circlearrowleft$$

$$\partial(p) := \sum_{\substack{q,\beta \\ \mathrm{ind}(\beta)=1}} \#\left(\mathcal{M}'(p,q)_\beta / \mathbb{R}\right) T^{\omega(\beta)} q$$

There is an isomorphism

$$\mathcal{M}(p,q) \longrightarrow \mathcal{M}'(p,q)$$

$$u(s,t) \longmapsto \varphi^{1-t}(u(s,t))$$

where φ^t = time-t flow of X_H, so we get

$$HF^*(L_0, L_1; H, J) \cong HF^*(\varphi(L_0), L_1; 0, J)$$

also (i.e., ∂'s match up because they're counting the same things).

Now let J_s, $s \in \mathbb{R}$, be a 1-parameter family of almost-complex structures with

$$J_s = \begin{cases} J & \text{for } s << 0 \\ J' & \text{for } s >> 0 \end{cases}$$

and let H_s be a family of functions with

$$H_s = \begin{cases} H & \text{for } s << 0 \\ H' & \text{for } s >> 0. \end{cases}$$

Consider a new moduli space:
for $p \in CF(L_0, L_1)$, $q \in CF(\varphi(L_0), L_1)$,

$$\mathcal{N}(p,q) = \left\{ \begin{array}{l} u: \mathbb{R} \times [0,1] \to M, \text{ s.t.} \\ \frac{\partial u}{\partial s} = J_s \left(\frac{\partial u}{\partial t} - X_{H_s} \right) \\ u(s,0) \in L_0 \\ u(s,1) \in L_1 \\ \lim_{s \to -\infty} u(s,t) = p(t) \\ \lim_{s \to +\infty} u(s,t) = q(t) \end{array} \right\}$$

$\mathbb{R}$ no longer acts on this moduli space, because of the dependence on s.

We define

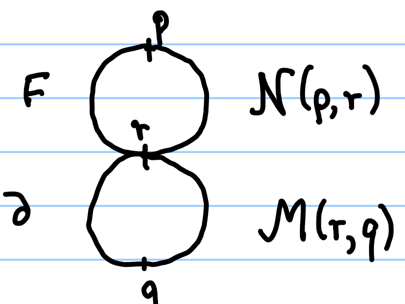
$$F: CF^*(L_0, L_1; H, J) \to CF^*(L_0, L_1; H', J')$$

by

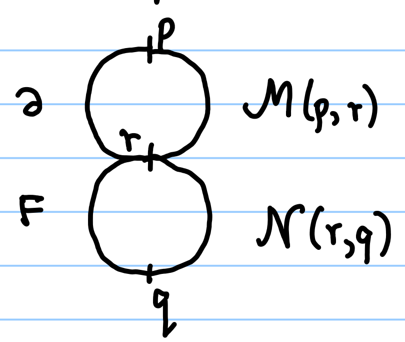
$$F(p) := \sum_{\substack{q,\beta: \\ \mathrm{ind}(\beta)=0}} \# \mathcal{N}(p,q)_\beta \, T^{\omega(\beta)} \, q.$$

This is a chain map: if we look at $\mathcal{N}(p,q)_\beta$ for $\mathrm{ind}(\beta) = 1$, then Gromov compactness says, if you add in nodal disks:

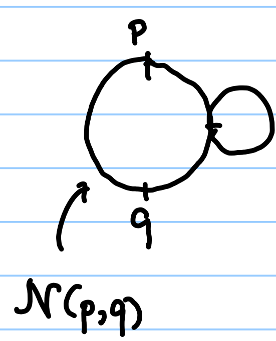
energy concentrates near:



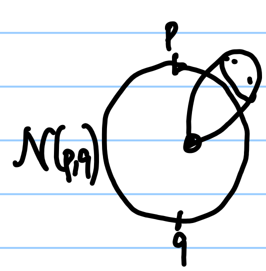
$s = +\infty$



$s = -\infty$



boundary point



interior point.

If we impose $[\omega] \cdot \pi_2(M, L_i) = 0$
to rule out the last two configurations, then

count of boundary points of compact 1-dim manifold is 0

$\Rightarrow F \circ \partial = \partial \circ F$

so F is a chain map.

Using similar arguments, you can show that

- F is independent of the choice of H_s, J_s, up to homotopy;

- $$\begin{tikzcd} CF^*(L_0, L_1; H, J) \arrow[r] \arrow[rr, bend right] & CF^*(L_0, L_1; H', J') \arrow[r] & CF^*(L_0, L_1; H'', J'') \end{tikzcd}$$

commutes, up to homotopy;

- $CF^*(L_0, L_1; H, J) \longrightarrow CF^*(L_0, L_1; H, J)$
  is the identity, if we choose
  $H_s = H$ = constant
  $J_s = J$ = constant.

This concludes the proof of the theorem.

Now, what is $HF^*(L,L)$? By the above theorem, we should define it to be $HF^*(\varphi_H(L), L)$, for some Hamiltonian H so that $\varphi_H(L) \pitchfork L$.

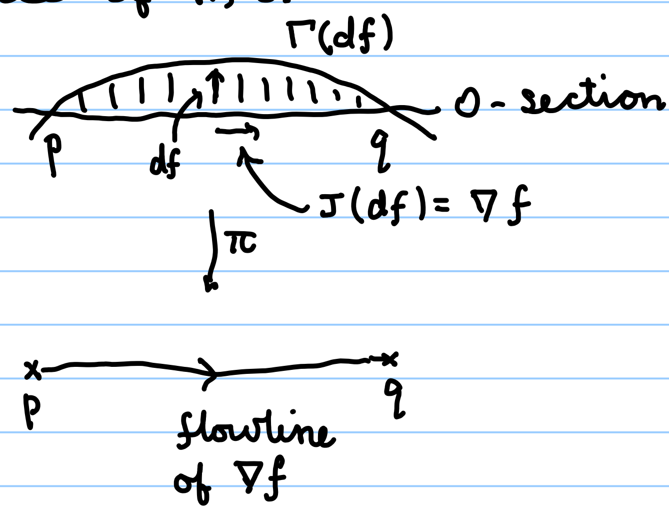
E.g. Let $M = T^*L$. This has a natural symplectic structure, w.r.t. which $L = 0$-section is Lagrangian.
If we take f to be some Morse function on L, then set $H = \pi^* f$ ($\pi : T^*L \to L$ projection), then

$$\varphi_H(L) = \{\text{graph of } df\} \subset T^*N.$$

In particular,

$$\{\varphi_H(L) \cap L\} = \{\text{crit. points of } f\}.$$

Furthermore, it turns out that the moduli space of J-holomorphic strips from p to q is <u>isomorphic</u> to the moduli space of Morse flowlines from p to q, for appropriate choice of H, J.



It follows that

$$HF^*(L,L) \cong H^*(L).$$

Weinstein's Lagrangian neighbourhood theorem: any lagrangian $L \subset M$ has a neighbourhood symplectomorphic to a neighbourhood of $L \subset T^*L$.

It follows that $CF^*(L,L) \cong C^*_{\text{Morse}}(L)$.

Furthermore, the <u>low-energy</u> strips in the differential correspond to Morse flowlines; so the filtration of $CF^*(L,L)$ by energy (= power of T in Λ) induces a spectral sequence

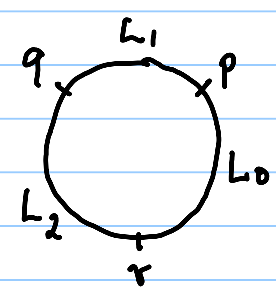
$$E_2 = H^*(L) \Rightarrow HF^*(L,L).$$

the <u>Oh spectral sequence</u>.

Now we define a <u>product</u>:

$$\mu^2 : HF^*(L_0, L_1) \otimes HF^*(L_1, L_2) \to HF^*(L_0, L_2)$$

by counting J-holomorphic <u>triangles</u>:

$\mathcal{M}(p, q, r) := \Big\{$ J-hol maps  $\Big\} \big/ \sim$

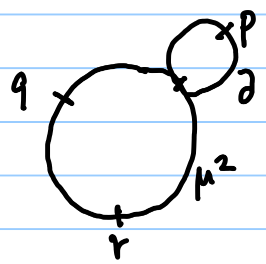
up to reparametrisation

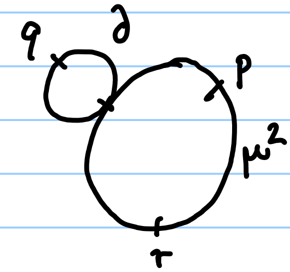
For $p \in CF^*(L_0, L_1)$, $q \in CF^*(L_1, L_2)$, define

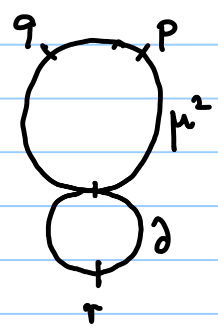
$$\mu^2(p,q) := \sum_{\substack{r \in CF^*(L_0,L_2) \\ \beta : \mathrm{ind}(\beta)=0}} \#\left(\mathcal{M}(p,q,r)_\beta\right) T^{\omega(\beta)}\, r$$

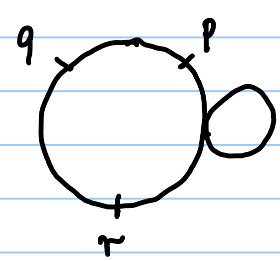
once again, Gromov compactness ensures the counts are finite for each β.

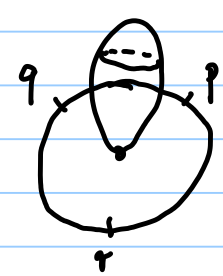
Now look at $\mathcal{M}(p,q,r)_\beta$ for $\mathrm{ind}(\beta)=1$. Its compactification $\overline{\mathcal{M}}(p,q,r)_\beta$ includes











etc.

Once again, under appropriate assumptions (e.g. $[\omega]\cdot\pi_2(M, L_i) = 0$), only the first 3 terms appear; so

$$\pm \mu^2(\partial p, q) \pm \mu^2(p, \partial q) \pm \partial\mu^2(p, q) = 0$$

$\Rightarrow$ μ^2 descends to a well-defined product on Floer cohomology,

$$[\mu^2] : HF^*(L_0, L_1) \otimes HF^*(L_1, L_2) \to HF^*(L_0, L_2).$$

If $L \subset T^*L$ is the 0-section,

$$[\mu^2] : HF^*(L, L) \otimes HF^*(L, L) \to HF^*(L, L)$$

is the cup product on Morse cohomology.