Open strings

Note Title — 2/9/2014

Last time, we saw

Operad	name	A-model	B-model
$H_*(\overline{M}_{0,n})$	CohFT	$QH^*_{big}(M)$	(Barann-Kont.)
$H_*(\mathrm{Conf}(\mathbb{C},n))$	Gerstenhaber algebra	$QH^*_{small}(M)$	$HT^*(M)$ $HT^*(M,W)$
$H_*(F\mathrm{Conf}(\mathbb{C},n))$	BV algebra	$SH^*(M)$ (M non-compact)	Same as if M admits hol. vol. form

Now, let's see another way of making Gerstenhaber and BV algebras.

Let A be a vector space. Define

$$CC^p(A)^s := \bigoplus_{s+t=p} \hom(A^{\otimes s}, A)^t$$

Define

$$\alpha \circ \beta(a_1, \dots, a_s) := \sum_{i,j} (-1)^* \alpha(a_1 \dots \quad \beta(a_{i+1} \dots) a_{i+j+1} \dots a_s)$$

$$* = |\beta|' (|a_1|' + \dots + |a_i|')$$

$$[\alpha, \beta] := \alpha \circ \beta - (-1)^{|\alpha|'|\beta|'} \beta \circ \alpha$$

(the Gerstenhaber bracket)

This makes $CC^*(A)$ into a graded Lie algebra.

Now suppose we equip A with an associative algebra structure

$$\bullet : A \otimes A \to A,$$

Define $\mu \in CC^2(A)$, $\mu(a,b) := (-1)^{|a|} a \bullet b$

Then

$$\begin{aligned}
\mu \circ \mu &= \mu(\mu(a,b),c) + (-1)^{|a|'} \mu(a, \mu(b,c)) \\
&= (-1)^{|a|+|a|+|b|} (a \bullet b) \bullet c + (-1)^{|a|'+|a|+|b|} a \bullet (b \bullet c) \\
&= (a \cdot b) \cdot c - a \cdot (b \cdot c) \\
&= 0.
\end{aligned}$$

$\Rightarrow [\mu, \mu] = 0,$

It follows that $\delta := [\mu, -] : CC^p(A) \to CC^{p+1}(A)$ satisfies $\delta^2 = 0$,

<u>Defn</u>: $HH^*(A) := H^*(CC^*(A), \delta)$ is the <u>Hochschild cohomology</u> of A.

<u>Thm</u> (Gerstenhaber): $HH^*(A)$ has a natural structure of Gerstenhaber algebra.

The Gerstenhaber bracket descends to cohomology, as $[\mu, -]$ satisfies a Leibniz rule, by the Jacobi identity.

The product on $HH^*(A)$ is induced by

$$\alpha \cup \beta (a_1, \dots, a_s) := (-1)^* \mu(\alpha(a_1 \dots a_i), \beta(a_{i+1}, \dots, a_s))$$

$$* = |\beta|'(|a_1|' + \dots + |a_i|').$$

Thm (Kaufmann): If A is a Frobenius algebra, i.e., comes equipped with a symmetric, nondegenerate inner product $\langle , \rangle$ which is invariant in the sense that

$$\langle \alpha \bullet \beta, \gamma \rangle = \langle \alpha, \beta \bullet \gamma \rangle,$$

then $HH^*(A)$ can be equipped with a natural structure of BV algebra, extending the above Gerstenhaber algebra structure.

More generally, let

$$\mu \in CC^2(A)^{\geq 1}$$

be an element satisfying $\mu \circ \mu = 0$. This is called an A_∞ algebra. Let's unpackage the definition:

We don't just have a product operation

$$\mu : A \otimes A \to A,$$

we have operations

$$\mu^s : A^{\otimes s} \to A$$

of degree $2-s$, for all $s \geq 1$.

We don't just have associativity of the product; we have a whole series of 'associativity' equations (called the 'A_∞ relations'):

① $\mu^1(\mu^1(a)) = 0 \Rightarrow \mu^1$ is a differential

Denote $H^*(A) := H^*(A, \mu^1)$.

② $\mu^1(\mu^2(a,b)) = \mu^2(\mu^1(a), b) + (-1)^{|a|'}\mu^2(a, \mu^1(b))$.

$\Rightarrow \mu^2$ descends to cohomology:

$$[\mu^2] : H^*(A)^{\otimes 2} \longrightarrow H^*(A).$$

③ $\mu^2(\mu^2(a,b),c) + (-1)^{|a|'}\mu^2(a, \mu^2(b,c))$

$+ \mu^1(\mu^3(a,b,c)) + \mu^3(\mu^1(a), b, c)$

$+ (-1)^{|a|'}\mu^3(a, \mu^1(b), c) + (-1)^{|a|'+|b|'}\mu^3(a,b,\mu^1(c)) = 0.$

$\Rightarrow [\mu^2]$ defines an associative product on $H^*(A)$.

④ ...

Think of an A_∞ algebra as being 'associative up to homotopy'.

Now, given an A_∞ algebra (A, μ),

$\mu \circ \mu = 0 \Rightarrow [\mu,\mu] = 0$.

$$\Rightarrow \delta : CC^*(A) \longrightarrow CC^{*+1}(A)$$

$$\delta(\alpha) = [\mu, \alpha]$$

satisfies $\delta^2 = 0$.

Define $HH^*(A) := H^*(CC^*(A), \delta)$, the Hochschild cohomology of the A_∞ algebra (A, μ). The Gerstenhaber bracket descends to cohomology as before, making $HH^*(A)$ a graded Lie algebra. The product

$$\alpha \cup \beta\,(a_1, \dots, a_s) :=$$

$$\sum_{i \le j \le k \le l} (-1)^* \mu(a_1, \dots, \alpha(a_{i+1}, \dots), a_{j+1}, \dots, \beta(a_{k+1}, \dots), a_{l+1}, \dots)$$

$$* = |\alpha|'(|a_1|' + \dots + |a_i|') + |\beta|'(|a_1|' + \dots + |a_k|')$$

is associative, and equips $HH^*(A)$ with the structure of a Gerstenhaber algebra.

Ultimately, we're going to see

Open strings $\qquad\qquad$ Closed strings

A_∞ algebra/category $\xrightarrow{HH^*}$ Gerstenhaber/BV algebra

Fukaya category $\longrightarrow$ QH^*, SH^*

$D^b\,Coh$ $\xrightarrow{HKR}$ HT^*

Now, let's see a geometric picture of the A_∞ operad.

For $k \ge 2$, let $R_k := \{$ configurations of $k+1$ points on the boundary of $\mathbb{D}$, $z_0, z_1, \dots, z_k$, in order $\}/\sim$

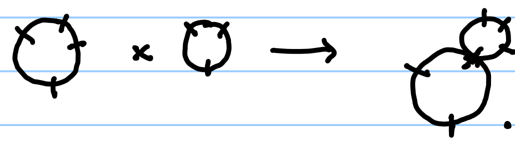
($\mathbb{D}$ = unit disk $\subset \mathbb{C}$), where $\sim$ denotes biholomorphism of the disk.

Let $\bar{\mathcal{R}}_k$ denote the Deligne-Mumford compactification of $\mathcal{R}_k$, by nodal disks.

$\bar{\mathcal{R}}_k$ are polyhedra, called <u>associahedra</u>,

and there are obvious maps

$$\circ_i : \bar{\mathcal{R}}_j \times \bar{\mathcal{R}}_k \longrightarrow \bar{\mathcal{R}}_{j+k-1}$$



We define a <u>DG operad</u> (DG = Differential Graded), by

$$\mathcal{P}(k) := C_*^{cell}(\bar{\mathcal{R}}_k).$$ chain complex; hence the 'DG'

with composition maps induced by the above $\circ_i$.

A DG operad is the same as an operad, but everything is a chain complex rather than a vector space. A DG algebra over a DG operad is then a chain complex (A, d), with maps of chain complexes

$$\mathcal{P}(k) \longrightarrow \mathrm{Hom}_{Kom}\left((A,d)^{\otimes k}, (A,d)\right),$$

etc.

Theorem: DG algebras over the DG operad $C_*(\bar{\mathcal{R}}_k)$ are equivalent to A_∞ algebras.

Pf: Given (A, d), the differential becomes μ^1, and the fundamental class of $\bar{\mathcal{R}}_k$ becomes μ^k.
The A_∞ relations follow because

$$d[\bar{\mathcal{R}}_s] = \sum_{i,\, j\geq 2,\, k\geq 2} [\bar{\mathcal{R}}_j] \circ_i [\bar{\mathcal{R}}_k]$$

(codimension-1 boundary of $\bar{\mathcal{R}}_s$ is given by disks with a single node)

$$\sum_{\substack{j \text{ or } k \\ =1}} \mu^j(\ldots \mu^k(\ldots)\ldots) = \sum_{\substack{j\geq 2 \\ k\geq 2}} \mu^j(\ldots \mu^k(\ldots)\ldots).$$

The fact that the moduli space of disks has codimension-1 boundary forces us to work with DG operads, which means all open-string invariants are defined 'up to homotopy', which gives them a different flavour to closed-string invariants.

Lagrangian Floer cohomology

Let (M, ω) be a symplectic manifold.
A submanifold

$$N \subset M$$

is called *isotropic* if $\omega|_N = 0$

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Isotropic submanifolds have dimension $\leq \frac{1}{2}\dim M$ (because ω is nondegenerate).

If $L \subset M$ is isotropic and $\dim L = \frac{1}{2}\dim M$, L is called <u>Lagrangian</u>. Lagrangian submanifolds are central to symplectic topology:

<u>Weinstein's creed</u>: "Everything is a Lagrangian".

Suppose $L_0, L_1 \subset (M,\omega)$ are compact Lagrangian submanifolds, with transverse intersections (= finite set of points, as both are half-dimensional).

Define the Floer cochain group,

$$CF^*(L_0, L_1) := \Lambda\langle L_0 \cap L_1 \rangle$$

Λ-vec. space with basis $= L_0 \cap L_1$,

where $\Lambda :=$ Novikov field

$$= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lambda_i \longrightarrow +\infty \right\}.$$

By putting some extra structure on the L_i, we can equip $CF^*(L_0, L_1)$ with a

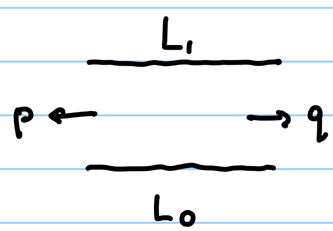
$\mathbb{Z}/2N$ - grading,

$N =$ minimal Chern number of M.

Let $J \in \mathrm{End}(TM)$, $J^2 = -\mathbb{1}$ be an almost-complex structure compatible with ω.

Given $p, q \in L_0 \cap L_1$, define

$$\mathcal{M}(p;q) := \Big\{ u : \mathbb{R} \times [0,1] \longrightarrow M$$
$$Du \circ j = J \circ Du \quad J\text{-holomorphic}$$
$$u(s,0) \in L_0$$
$$u(s,1) \in L_1$$
$$\lim_{s \to +\infty} u(s,t) = p$$
$$\lim_{s \to -\infty} u(s,t) = q \Big\}$$



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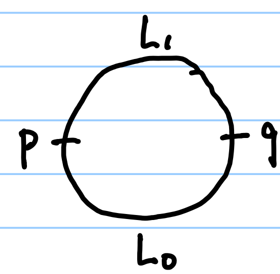

There's an $\mathbb{R}$-action on this moduli space,

$c: \quad u(s,t) \longmapsto u(s+c, t)$.

Let $\mathcal{M}(p;q)_\beta :=$ maps in homology class $\beta \in \pi_2(M; L_0, L_1)$.

This is a <u>Fredholm problem</u>, so $\mathcal{M}(p;q)_\beta$ is generically a smooth manifold. Its dimension is given by the <u>Maslov index</u> $\mathrm{ind}(\beta)$ (comes from $\pi_1(\mathcal{L}Gr) \cong \mathbb{Z}$)

space of Lagrangian subspaces of a symplectic vector space.