BV algebras and symplectic cohomology

Note Title 12/9/2013

Recall:

We have seen the operads

	Name	A-model	B-model
$H_*(\overline{\mathcal{M}}_{0,k+1})$ $\uparrow$ $H_*(\mathcal{M}_{0,k+1})$ $\uparrow$	Cohomological Field Theory	GW invts; 'big' quantum cohomology $QH^*(M)$	? [BK]
$H_*(\mathrm{Conf}(\mathbb{C},k))$ $\uparrow \downarrow$	Gerstenhaber algebra	?	$HT^*(M)$, $HT^*(M,W)$
$H_*(\mathrm{FConf}(\mathbb{C},k))$	Batalin-Vilkovisky (BV) algebra	? $SH^*(M)$ symplectic cohomology	$HT^*(M)$, if M admits a holom. vol. form

today

Recall a Gerstenhaber algebra $(A, \bullet, [,])$ has an associative product $\bullet$, Lie bracket $[,]$, and they're compatible in a certain way.

Defn: A Batalin-Vilkovisky (BV) algebra is a Gerstenhaber algebra $(A, \bullet, [,])$ together with an additional operation

$$\Delta : A \longrightarrow A,$$

satisfying:

$$|\Delta(a)| = |a| - 1;$$

$$\Delta^2 = 0 ;$$

$$(*) \qquad [a,b] = (-1)^{|a|}\Delta(a\cdot b) - (-1)^{|a|}\Delta(a)\cdot b - a\cdot\Delta(b).$$

E.g. We saw last time that, if M has a holomorphic volume form $\Omega \in \Omega^{n,0}(M)$, then we can define

$$\Delta : HT^*(M) \longrightarrow HT^{*-1}(M)$$

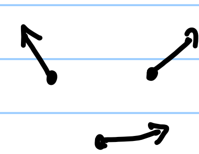
by pulling back $\partial : H^*(\mathcal{A}^{p,q}(M), \bar\partial) \circlearrowleft$ via the isomorphism $HT^*(M) \xrightarrow[\lrcorner\Omega]{} H^*(\Omega M)$.

It satisfies $\Delta^2 = 0$ clearly, and the Tian-Todorov lemma says it also satisfies $(*)$.

So if M is Calabi-Yau, $HT^*(M)$ is a BV algebra.

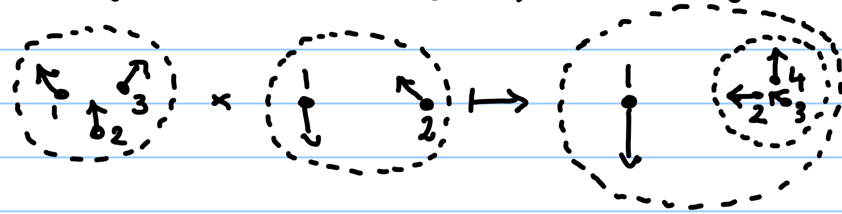
If M is also compact Kähler, $\partial = 0$ on cohomology (Hodge to de Rham degenerates), so $\Delta = 0 \Rightarrow [\,,\,]_{SN} = 0$.

Define $FConf(\mathbb{C}, k)$ (framed configuration space) to be the space of configurations of k points in $\mathbb{C}$, where each point comes with a choice of 'framing', i.e., a chosen direction:



We have composition operators as before, but now we rotate when we insert:

$\circ_2 : FConf(\mathbb{C}, 3) \otimes FConf(\mathbb{C}, 2) \to FConf(\mathbb{C}, 4)$



<u>Thm</u> (Getzler): Algebras over the operad

$$\begin{cases} \mathcal{P}(1) = \tilde{H}_*(FConf(\mathbb{C}, 1)) \\ \mathcal{P}(m) = H_*(FConf(\mathbb{C}, m)) \qquad m \geq 2 \end{cases}$$

are equivalent to BV algebras.

<u>Idea:</u> Δ corresponds to the class in $H_1(FConf(\mathbb{C}, 1))$:

There is an obvious inclusion

$$Conf(\mathbb{C}, k) \hookrightarrow FConf(\mathbb{C}, k) \to Conf(\mathbb{C}, k)$$

(make all the arrows point down), so we obtain the Gerstenhaber structure. To see $(*)$, observe that $FConf(\mathbb{C}, 2) \cong (S^1)^3 \times \mathbb{R}^2 \times \mathbb{R}_+$; the LHS corresponds to the diagonal class $\theta_1 + \theta_2 + \theta_3$ in $H_1((S^1)^3)$, whereas the terms on the RHS correspond to $\theta_1, \theta_2, \theta_3$ respectively.

Now observe that:

On B-model, we get a Gerstenhaber algebra $HT^*(M)$ (or $HT^*(M,W)$). If M is Calabi-Yau then $HT^*(M)$ is a BV algebra; if M is furthermore compact, then the BV operator is 0 (hence so is the Lie bracket). But if M is noncompact, the Lie bracket might be nonzero:

E.g. $M^\vee = \mathbb{C}^*$, $HT^*(M^\vee) \cong \mathbb{C}[z, z^{-1}, \partial_z]$, and

$$[z, \partial_z] = \partial_z(z) = 1.$$

$$\Omega = z^{-1} dz \Rightarrow \Delta(z^k) = 0, \quad \Delta(z^k \partial_z] = (k-1) z^{k-1}.$$

So what could be mirror to this in the A-model?

Answer: <u>symplectic cohomology</u>. This is an invariant of certain <u>noncompact</u> symplectic manifolds called <u>Liouville manifolds</u>.

<u>Defn</u>: A Liouville <u>domain</u> is a compact symplectic mfd with boundary (M,ω), equipped with a Liouville 1-form θ:

$$\omega = d\theta$$

so that the associated Liouville vector field Z:

$$\iota_Z \omega = \theta$$

points outwards along the boundary ∂M.

This means $\theta|_{\partial M}$ is a contact 1-form.

A Liouville domain M can be <u>completed</u> to a non-compact symplectic manifold $\hat{M}$ by attaching a copy of $\partial M \times [0, \infty)$ to ∂M, with symplectic form $\theta|_{\partial M} \wedge dr$.

Noncompact symplectic manifolds obtained in this way are called Liouville manifolds (of finite type).

E.g. Smooth complex affine varieties can be given a Liouville manifold structure.

E.g. T^*L, for any smooth compact mfd, is a Liouville manifold.

Now if M is a Liouville domain, $\theta|_{\partial M}$ is a contact form; the associated <u>Reeb vector field</u> R is characterised by:

- $d\theta|_M (R, -) = 0$
- $\theta|_M (R) = 1$.

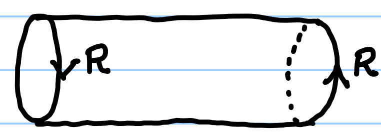
If $\hat{M}$ is the completion of a Liouville domain M, then $SH^*(\hat{M})$, the symplectic cohomology of $\hat{M}$, is the cohomology of a chain complex $(SC^*(\hat{M}), d)$, with generators:

- $C^*(M)$, cochain complex of M
- for each orbit of the Reeb vector field γ, two generators, $\hat{\gamma}$ and $\check{\gamma}$.

(assuming orbits of R are isolated, as is generically the case).

<u>Thm</u>: $SH^*(\hat{M})$ naturally has the structure of a BV algebra.

E.g. $\hat{M} = \mathbb{C}^*$ is completion of cylinder:



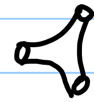
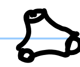
$\rightsquigarrow$ generators of $SC^*(\mathbb{C}^*) \overset{(d=0)}{\cong} SH^*(\mathbb{C}^*)$ are

$$\begin{array}{ccccccc} & & & H^*(\mathbb{C}^*) & & & \\ & z^{-1}\partial_z & \partial_z & z\partial_z & z^2\partial_z & z^3\partial_z & \\ \cdots & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\ & z^{-2} & z^{-1} & 1 & z & z^2 & \end{array}$$

In fact, $\mathbb{C}^*$ is mirror to itself, and so there is an isomorphism

$$SH^*(\mathbb{C}^*) \cong HT^*(\mathbb{C}^*)$$

as BV algebras.

Ex: The pair of pants  is mirror to the L-G model $(\mathbb{C}^3, W = xyz)$. Check there's a natural correspondence between generators of $SH^*($$)$ and those of $\mathrm{Jac}(W)$.

Now it's time to say more about the definition of $SH^*(\hat{M})$.

Hamiltonian Floer homology:

There is another formulation of the closed-string A-model, which makes the reason for the terminology 'closed-string' more evident, and is more general.

First, a reminder about Morse cohomology:

$M =$ manifold

$f : M \to \mathbb{R}$ smooth, nondeg. crit. points

$g =$ Riem. metric

$CM^*(f) := \mathbb{C}\langle \text{crit } f \rangle$ graded by Morse index.

Given $p, q \in \text{crit } f$, $\mathcal{M}(p,q) := \{ u : \mathbb{R} \to M$

$\dot{u} = \nabla f,\ u(+\infty) = p,\ u(-\infty) = q \} / \mathbb{R}$

Generically a smooth mfd of dimension

$\dim \mathcal{M}(p,q) = i(q) - i(p) - 1$

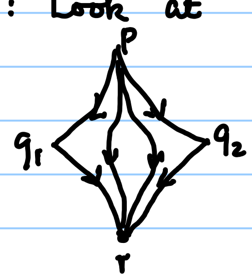
$d_M : CM^*(f) \to CM^{*+1}(f)$

$$\langle d_M p, q \rangle := \#(\mathcal{M}(p,q))$$

means count 0-dim component.

Lem: $d_M^2 = 0$

Pf: Look at 1-dim moduli space:



$$\Rightarrow \sum_q \langle d_M p, q \rangle \langle d_M q, r \rangle = 0$$

$$\Rightarrow \langle d_M^2 p, r \rangle = 0.$$

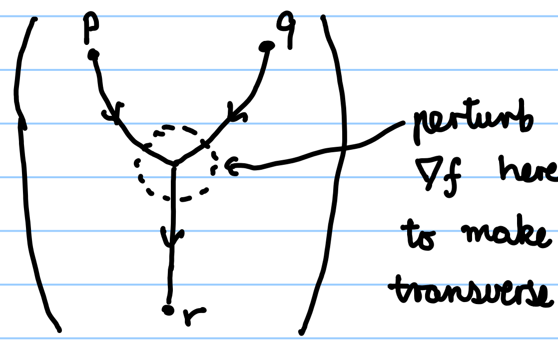
$$H^*(CM^*(f), d_M) \cong H^*(M).$$

The map is given by

$$p \longmapsto \text{P.D.}(\text{ascending manifold from } p).$$

From this, one can see that the cup product on $CM^*(f)$ is given by

$$\langle p \smile q, r \rangle := \#$$



__Now, the A-model:__

We consider the based loop space of our manifold:

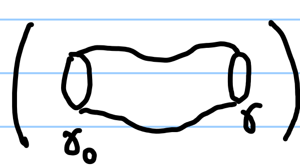
$$\Omega M := C^\infty(S^1, M).$$

If the symplectic form ω were exact, $\omega = d\theta$, we could define a __symplectic action functional__ on ΩM:

$$\mathcal{A}(\gamma) = \int_\gamma \theta.$$

(if ω not exact, $\mathcal{A}$ is only defined up to adding a constant: near γ_0,

$$\mathcal{A}(\gamma) := \omega$$



so the 1-form $d\mathcal{A}$ is still well-defined).

We try to calculate Morse cohomology of ΩM, using the 'Morse function' $\mathcal{A}$.
We find that the critical points of $\mathcal{A}$ are the constant loops: so they are not isolated, and we perturb H.
We do this by choosing a function

$$H : S^1 \times M \longrightarrow \mathbb{R},$$

and considering the action functional

$$\mathcal{A}_H(\gamma) := \int_\gamma \theta + dH.$$

Result: $\mathrm{crit}(\mathcal{A}_H)$ = time-1 orbits of the corresponding <u>Hamiltonian vector field</u> X_H, which is defined by

$$\omega(X_H, \cdot) = dH.$$

Now $T_\gamma \Omega M \cong \Gamma(\gamma^* TM)$

Any a.c. structure J provides a metric:

$$\langle \xi, \eta \rangle = \int_0^1 \omega(J\xi(t), \eta(t))\, dt$$

The Morse flow equation then becomes:

for $\gamma_\pm \in \mathrm{crit}(\mathcal{A}_H)$,

$$\mathcal{M}(\gamma_+,\gamma_-) = \Big\{ u : \mathbb{R} \longrightarrow \Omega M \iff u : \mathbb{R}\times S^1 \longrightarrow M$$

$$\dot{u} = \nabla \mathcal{A}_H \iff \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H\right) = 0.$$

$$\lim_{s\to\pm\infty} u(s) = \gamma_\pm \iff \lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t). \Big\} \Big/ \mathbb{R}$$

under certain assumptions, these are smooth manifolds. They split up into homology classes β:

$$\mathcal{M}(\gamma_+,\gamma_-;\beta) = \{ u \text{ as above};\ [u]=\beta \}.$$

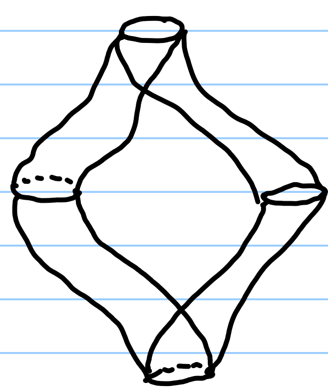
Now define $CF^*(H) := \Lambda\langle \mathrm{crit}\,\mathcal{A}_H \rangle$, and

$$\langle d\gamma_+, \gamma_- \rangle = \sum_\beta \#\mathcal{M}(\gamma_+,\gamma_-;\beta)\, T^{\omega(\beta)}$$

count 0-dim part of moduli space

Lem: $d^2 = 0$.

Pf: Look at 1-dim moduli space:



adding 'broken Floer trajectories' makes our moduli space compact (c.f. Gromov compactness: need to rule out things like

).

Argument runs as in Morse theory.

Defn: $HF^*(H) := H^*(CF^*(H), d)$.

Thm (Piunikhin-Salamon-Schwarz): If M is compact and monotone/Calabi-Yau (the cases of interest in mirror symmetry), then

$$HF^*(H) \cong H^*(M; \Lambda).$$

Reason: First show $HF^*(H)$ is independent of H (proof analogous to Morse cohomology). Then choose H small, time-independent; then only time-1 orbits of H are constant orbits at critical points; compare with Morse cohomology.

Rmk: $HF^*(H)$ naturally should have the structure of a BV algebra.

Pf: We consider the moduli space $FConf(\mathbb{C}, k)$. For each element of it, consider it as $\mathbb{CP}^1 \setminus \{\infty\}$ with marked points $\{z_1, \dots, z_k\}$. Choose holomorphic embeddings

$$\epsilon_j : \mathbb{D} \to \mathbb{C}, \quad \epsilon_j(0) = z_j,$$

$$\epsilon_j(\mathbb{R}_+) = \text{the 'framing' direction},$$

and

$$\epsilon_0 : \mathbb{D} \to \mathbb{CP}^1, \quad \epsilon_0(0) = \infty$$
$$\epsilon_0(\mathbb{R}_+) = \mathbb{R}_+.$$

Then furthermore choose, for each element,

$$K \in \Omega^1(\mathbb{C} \setminus \{z_1, \dots, z_k\}, C^\infty(M)),$$

so that

$$\epsilon_j^* K = H(\theta)\, d\theta, \quad \epsilon_0^* K = H(-\theta)\, d(-\theta).$$

Then for any orbits $\gamma_0, \gamma_1, \ldots, \gamma_k$ of H, we consider

$$\mathcal{M}(\gamma_0, \gamma_1, \ldots, \gamma_k) := \{ (r, u) : r \in FConf(\mathbb{C}, k);$$

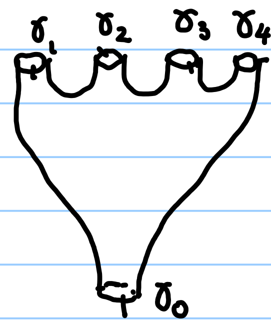
$$u : \mathbb{C} \setminus \{z_1, \ldots, z_k\} \longrightarrow M;$$

$(Du - Y)^{0,1} = 0$, where $Y : T\mathbb{C} \to TM$ is defined by

$Y(\xi) :=$ Ham. vector field assoc. to $K(\xi)$;

$$\lim_{r \to 0} u(\epsilon_j(re^{i\theta})) = \gamma_j(\theta) \quad (\gamma_0(-\theta) \text{ for } j = 0) \}$$

Picture:



Our generators are <u>parametrised</u> loops, so we really need $FConf(\mathbb{C}, k)$ to make sense of this.

As for GW invariants, there's a map

$$\mathcal{M}(\gamma_0; \gamma_1, \ldots, \gamma_k) \longrightarrow FConf(\mathbb{C}, k)$$

forgetting the map u; and it follows that $HF^*(H)$ is an algebra over the operad $H_*(FConf(\mathbb{C}, k))$.

Given a homology class represented by a cycle C in $FConf(\mathbb{C}, k)$, the corresponding operation is (morally) given by

$$\Phi_C : HF^*(H)^{\otimes k} \longrightarrow HF^*(H),$$

$$\langle \Phi_C(\gamma_1 \otimes \ldots \otimes \gamma_k), \gamma_0 \rangle = \#\{(r,u) \in \mathcal{M}(\gamma_1, \ldots, \gamma_k ; \gamma_0) : r \in C\}.$$

(in reality, only need to construct d, Δ, product, and check compatibility: technically easier, and loses no info from what you expect theory to contain!).

Now suppose $(M, \omega = d\theta)$ is a Liouville domain, with completion $\hat{M}$. $SH^*(\hat{M})$ is defined to be $HF^*(H)$, where

$$H : S^1 \times \hat{M} \longrightarrow \mathbb{R}$$

$$H(\theta, (r, m)) = h(r) \quad \text{on } \mathbb{R} \times \partial M$$

where $h'(0) > 0$ small,
$h''(r) > 0$,
$h'(r) \to +\infty$ as $r \to +\infty$.

Then $X_H = h'(r) R$
$\uparrow$ Reeb vec. field, in ∂M direction

So 1-periodic orbits of X_H = __either__ critical points of H (gives Morse cohomology of M) __or__ orbits of Reeb vector field, appearing at height r, such that $h'(r)$ = period.

The orbits coming from Reeb orbits come in S^1-families (from choice of starting point on loop), so we must perturb H to make these transverse; the S^1 family breaks into two loops, corresponding to generators of $H^*(S^1)$.