BV algebras.

Note Title 11/12/2013

Last time we saw the operad $H_*(\mathrm{Conf}(\mathbb{C},k))$. Algebras over this algebra are Gerstenhaber algebras. A Gerstenhaber algebra is a graded (in $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{N}$, for us) vector space A, with operations

$$\bullet : A^{\otimes 2} \longrightarrow A, \quad [,] : A^{\otimes 2} \longrightarrow A,$$

satisfying

$$|a \bullet b| = |a| + |b| \qquad (|a| = \text{degree of pure element } a)$$

$$|[a,b]|' = |a|' + |b|' \qquad (|a|' = |a| - 1)$$

$$a(bc) = (ab)c$$

$$ab = (-1)^{|a||b|} ba$$

$$[a,b] = -(-1)^{|a|'|b|'}[b,a]$$

$$[a,bc] = [a,b]c + (-1)^{|a|'|b|} b[a,c]$$

$$(-1)^{|a|'|c|'}[[a,b],c] + (-1)^{|b|'|a|'}[[b,c],a] + (-1)^{|c|'|b|'}[[c,a],b] = 0$$

We saw the GW invariants make $H^*(M)$ into a G.A., but with trivial Lie bracket. We saw that if M is a complex manifold, or non-singular variety over a field $\Bbbk$,

$$HT^*(M) := H^*(\Lambda^* TM)$$

is a $\mathbb{Z}$-graded Gerstenhaber algebra ($\bullet = \wedge$, $[,] =$ Schouten-Nijenhuis bracket).

CORRECTION (thank you Chris!): If (M, W) is a Landau-Ginzburg model, then

$$(\Lambda^* TM, [W,-])$$

is a complex of sheaves, and

$$HT^*(M,W) := \mathbb{H}^*(\Lambda^* TM, [W,-])$$

is the hypercohomology of this complex (not the cohomology of $(H^*(\Lambda^* TM), [W,-])$ as I said last time: that's a page in the spectral sequence).

E.g. If M is a nonsingular variety/$\mathbb{C}$,

$$HT^*(M,W) \cong H^*(\underbrace{\mathcal{A}^{0,q}(\Lambda^* TM)}_{C^\infty\ (0,q) \text{ forms w/ vals in } \Lambda^* TM}, \bar\partial + \underbrace{[W,-]}_{=\iota_{dW}}).$$

It's a $\mathbb{Z}/2$-graded Gerstenhaber algebra.

E.g. If M is a nonsingular variety/$\mathbb{k}$, and if $W: M \to \mathbb{k}$ has isolated critical points, i.e. $\mathrm{Sing}(W)$ is 0-dimensional, then

$$\begin{array}{c} \mathcal{O} \xleftarrow{dW} TM \xleftarrow{[W,-]} \Lambda^2 TM \leftarrow \cdots \\ \downarrow \\ \mathcal{O}/\mathcal{I} \end{array}$$

$\mathcal{I} = \mathrm{im}(dW)$ a sheaf of ideals of $\mathrm{Sing}(W)$

is a quasi-isomorphism, so

$$HT^*(M,W) \cong H^0(\mathcal{O}/\mathcal{I}) \cong \bigoplus_{p \in \mathrm{Sing}\, W} \mathcal{O}_p / \mathcal{I}_p.$$

and $[,] = 0$, as it vanishes on $\mathcal{O}$.

E.g. $M = \mathrm{Spec}\, R$ is a smooth affine variety/$\mathbb{k}$, then

$$HT^*(M,W) \cong H(\Lambda^* \mathrm{Der}_{\mathbb{k}} R, \iota_{dW}).$$

Kodaira-Spencer maps (analytically)

Suppose $\begin{array}{c}\mathcal{M}\\ \pi\downarrow \\ B\end{array}$ is a family of smooth complex varieties. Then for each $b \in B$ we have $M_b := \pi^{-1}(b)$, and $HT^*(M_b)$.

Defn: The Kodaira-Spencer map is

$$K.S.: T_b B \longrightarrow H^1(TM_b) \subset HT^*(M_b),$$

defined by taking $v \in T_b B$, lifting it to a smooth vector field

$$\tilde{v} \in \mathcal{A}^{0,0}(TM),$$

then taking $\bar\partial \tilde{v} \in \mathcal{A}^{0,1}(TM)$; it's in the image of $\mathcal{A}^{0,1}(TM_b)$ because $\bar\partial v = 0$. We define $K.S.(v)$ to be its preimage; it's independent of the choice of $\tilde{v}$.

Alternatively, it's the first connecting homomorphism in the LES in cohomology coming from the SES

$$0 \to TM_b \to TM \to \pi^* TB \to 0.$$

If we have $W: \mathcal{M} \to \mathbb{C}$ holomorphic, then we have a family of L.G. models over B. We similarly define
$$KS: T_b B \to HT^*(M_b, W_b)$$

by $\quad KS(v) = \tilde{v}(W) \quad$ if M_b is affine,
$((\bar\partial + \iota_{dW})\tilde{v}$? perhaps?).
$\tilde{v}$ = a holomorphic lift of v.

Examples:

In general it is easier to prove mirror symmetry over $\Lambda := \Lambda_0[T^{-1}]$, which is a field, than over Λ_0; inverting
T means we stay away from the singular point in the moduli space where $T=0$.
(typically, the mirror is given over $\mathbb{C}[\![T]\!] \subset \Lambda$ and one obtains the thing mirror to the A-model, which is defined over Λ, by base extension).

M.S. predicts, if M mirror to $M^\vee/\mathrm{Spec}\,\Lambda$, then we have

isomorphism of Λ-algebras (or $\mathbb{C}$-algebras)

$$\begin{tikzcd} QH^*(M) \arrow[r, leftrightarrow, "\sim"] & HT^*(M^\vee) \\ \Lambda \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] \arrow[ur, phantom, "\circlearrowright"] & \Lambda \arrow[u, hook] \end{tikzcd}$$

automorphism: mirror map

think:
$$\begin{tikzcd} \mathcal{M}_{symp} \arrow[r, leftrightarrow, "\sim"] \arrow[d, equal] & \mathcal{M}_{cpx} \arrow[d, equal] \\ \mathrm{Spec}\,\Lambda \arrow[r, leftrightarrow, "\sim"] & \mathrm{Spec}\,\Lambda \end{tikzcd}$$

A general feature of M.S. (see first lecture):

$QH^*(M) \cong T\mathcal{M}^{ext}_{symp}$ $\uparrow$ $T(\mathrm{Spec}\,\Lambda)$	$HT^*(M^\vee) \cong T\mathcal{M}^{ext}_{cpx}$ $\uparrow$ $T(\mathrm{Spec}\,\Lambda)$
and $T\frac{\partial}{\partial T} \mapsto [\omega] \in QH^2(M)$	and $T\frac{\partial}{\partial T} \mapsto K.S.(T\frac{\partial}{\partial T})$, the Kodaira-Spencer map.

E.g. $M = \mathbb{T}^2$, $QH^*(M) \cong \Lambda^*(\Lambda^2)$ (no J-hol. spheres in $\mathbb{T}^2$, so quantum cup product = classical cup product).

$M^\vee$ = elliptic curve / Λ (think of a family of complex elliptic curves).

$$\begin{aligned}
HT^*(M^\vee) &\cong H^*(\mathcal{O} \oplus TM^\vee) \\
&\cong H^*\left(\mathcal{O}[\partial_z]/\partial_z^2\right) \quad \text{think analytic world.} \\
&\cong H^*(\mathcal{O}) \otimes_\Lambda \Lambda[\partial_z]/\partial_z^2 \\
&\cong \Lambda[d\bar{z}]/d\bar{z}^2 \otimes \Lambda[\partial_z]/\partial_z^2 \\
&\cong \Lambda^*(d\bar{z}, \partial_z).
\end{aligned}$$

($\cdot$ = ext. product, $[,] = 0$)

Doesn't have much content...

E.g. $M = \mathbb{CP}^1$, $QH^*(M) \cong \Lambda[z]/z^2 = T$

$$(M^\vee, W^\vee) = \left(\mathrm{Spec}(\Lambda[z, z^{-1}]), W^\vee = Tz + Tz^{-1}\right)$$

$$HT^*(M^\vee, W^\vee) \cong \Lambda[z, z^{-1}]/z^2 = 1 \cong \Lambda[z]/z^2 = 1$$

$$[,] = 0$$

$$K.S.\left(T\frac{\partial}{\partial T}\right) = T\frac{\partial}{\partial T}W^\vee = Tz + Tz^{-1} = 2Tz$$

We have

$$\begin{tikzcd}
QH^*(\mathbb{CP}^1) \cong \Lambda[z]/z^2 = T \arrow[r, leftrightarrow, "z \mapsto 2Tz"] & \Lambda[z]/z^2 = 1 \cong HT^*(M^\vee, W^\vee) \\
\Lambda \arrow[u, hook] \arrow[r, leftrightarrow] & \Lambda \arrow[u, hook] \\
T \arrow[r, mapsto] & 4T^2
\end{tikzcd}$$

E.g. $M =$ smooth degree-a hypersurface $\subset \mathbb{CP}^{n+1}$
with Fubini-Study form ω
(P.D. to hyperplane class), $a \le n+1$

$\widetilde{M}^\vee = \mathbb{A}^{n+2}_\Lambda = \operatorname{Spec}(\Lambda[u_1, \dots, u_{n+2}])$

$\widetilde{W}^\vee = -u_1 \dots u_{n+2} \quad + \sum_{i=1}^{n+2} T u_i^a.$

$\Gamma := \ker\left((\mathbb{Z}/a)^{n+2} \xrightarrow[\Sigma]{} \mathbb{Z}/a\right)$

acts on $\widetilde{M}^\vee$ by ath roots of unity
multiplying coordinate functions.

$M^\vee := \widetilde{M}^\vee / \Gamma$, $\widetilde{W}^\vee$ is Γ-invariant $\Rightarrow$
induces $W^\vee$ on $M^\vee$.

Won't say exact definition of $HT^*(M^\vee; W^\vee)$,
but
$$HT^*(M^\vee; W^\vee) \supset HT^*(\widetilde{M}^\vee, \widetilde{W}^\vee)^\Gamma$$
and this contains K.S.$(T\partial/\partial T)$.

$HT^*(\widetilde{M}^\vee, \widetilde{W}^\vee)^\Gamma \cong \operatorname{Jac}(\widetilde{W}^\vee)^\Gamma$

$$\cong \Lambda[u_1^a, \dots, u_4^a, u_1 \dots u_4] \Big/ \left\langle \frac{\partial \widetilde{W}^\vee}{\partial u_i} \right\rangle^\Gamma$$

$\dfrac{\partial W^\vee}{\partial u_i} = -u_1 \dots \hat{u}_i \dots u_{n+2} + a T u_i^{a-1} \quad (-a!\, T^{n+2}$
if $a = n+1)$

So in this ring,
$$u_1 \dots \hat{u}_i \dots u_{n+2} = a T u_i^{a-1}$$

$$\Rightarrow \quad u_1 \dots u_{n+2} = a T u_i^a$$

and $(u_1 \dots u_{n+2})^{n+1} = a^{n+2} T^{n+2} (u_1 \dots u_{n+2})^{a-1}$.

so $\mathrm{Jac}(\tilde{W}^\vee)^\Gamma \cong \Lambda[U] / U^{n+1} = a^{n+2} T^{n+2} U^{a-1}$

and $T \frac{\partial}{\partial T} \tilde{W}^\vee = \sum_{i=1}^{n+2} T u_i^a = \frac{n+2}{a} U$ $\left(- a!\,(n+2) T^{n+2} \text{ if } a = n+1 \right)$

If we regard $M := \{ \sum z_j^a = 0 \}$ then

$$\Gamma^* := \hom(\Gamma, \mathbb{C}^*) = (\mathbb{Z}/a)^{n+2} / (\mathbb{Z}/a)$$

acts on M by symplectomorphisms, and

$$\begin{array}{ccc} QH^*(M)^{\Gamma^*} & \cong & \mathrm{Jac}(\tilde{W}^\vee)^\Gamma \\ \cup & & \cup \\ \Lambda & \longrightarrow & \Lambda \\ T & \longmapsto & T^{n+2} \end{array}$$

M.S. predicts $[\omega] = $ P.D. (U coord divs) $= (n+2) D$, where $D =$ hyperplane class, corresponds to $\mathcal{K}$, so

$(n+2) D \longmapsto \mathcal{K}$

$\Rightarrow D \longmapsto \frac{1}{a} U$

$\Rightarrow D^{*(n+1)} = a^a T D^{*(a-1)}$ in $QH^*(M)$

$\left((D + a!\,T)^{n+1} = a^a T (D + a!\,T)^{a-1} \text{ if } a = n+1 \right)$.

E.g. $M =$ cubic surface, $a = 3 = n+1$

$$(D+6T)^3 = 27T(D+6T)^2$$

$$D^3 = 9TD^2 + 6^3T^2D + 27\cdot 28\,T^3$$

$$\Rightarrow \#\text{lines} = \langle D^{*2}, D\rangle$$
$$= \langle D^{*2}, D*1\rangle$$
$$= \langle D^{*3}, 1\rangle \quad \text{using } \langle \alpha*\beta, \gamma\rangle = \langle \alpha, \beta*\gamma\rangle$$
$$= \langle 9TD^2 + 6^3T^2D + 27\cdot 28\,T^3, 1\rangle$$
$$= \langle 9T\,D^{*2}, 1\rangle$$
$$= \langle 9TD, D\rangle$$
$$= 9T \times 3$$
$$= 27\,T.$$