Gerstenhaber and BV algebras

Note Title — 11/12/2013

<u>Defn</u>: Let

$$\mathrm{Conf}(\mathbb{C},k) := \{(z_1,\dots,z_k)\in\mathbb{C}^k : z_i \neq z_j \text{ for } i\neq j\}.$$

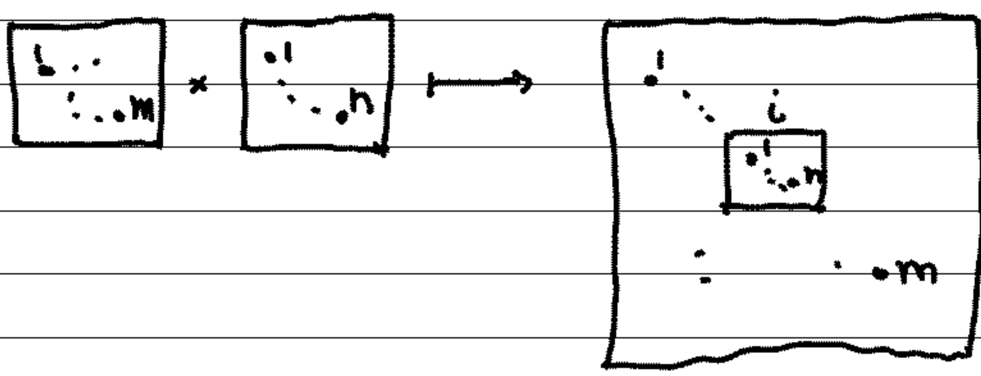
be the configuration space of k distinct points in $\mathbb{C}$.

We define an operad $\mathcal{P}$ with

$$\mathcal{P}(k) := H_*(\mathrm{Conf}(\mathbb{C},k)) \quad \text{for } k \geq 2.$$

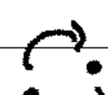
The composition operators are defined by

$$\circ_i : \mathcal{P}(m)\otimes\mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$$



'shrinking the configuration with n marked points and gluing it in at the ith of the m marked points'.

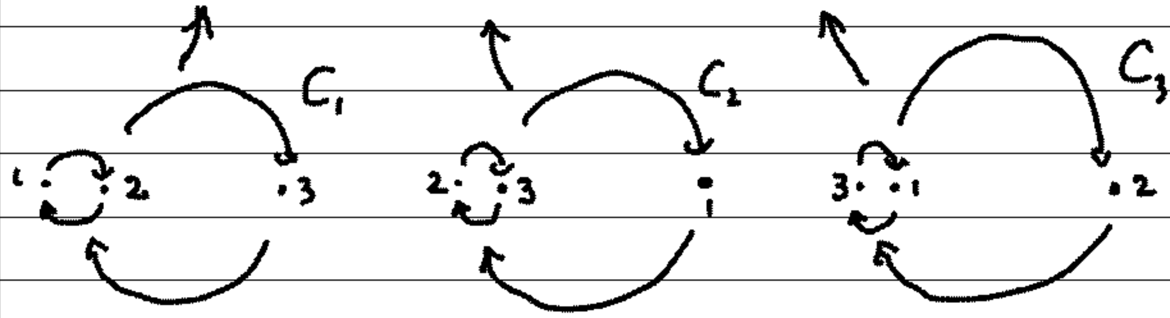
$$H_0(\mathrm{Conf}(\mathbb{C},2)) = k$$

$H_1(\mathrm{Conf}(\mathbb{C},2)) = k$, generated by  .

These two operations generate $\mathcal{P}$. The class in H_0 corresponds to a product $\bullet$, and the class in H_1 corresponds to a bracket $[,]$. The product is associative, by much the same argument as before.

The bracket satisfies the Jacobi identity. To see why, note that

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$$



$\cong$

$$\left\{ \begin{aligned} z_3 &= 0 \\ z_1 &= e^{i\theta_1} \\ z_2 &= e^{i\theta_1} + \varepsilon e^{i\theta_2} \end{aligned} \right\}$$

we must show

$$C_1 + C_2 + C_3 = 0$$

in $H_2(\mathrm{Conf}(\mathbb{C},3))$.

Define a 3-chain $S^1 \times S^1 \times I \longrightarrow \mathrm{Conf}(\mathbb{C},3)$ by

$$(\theta_1, \theta_2, t) \longmapsto z_3 = 0, \quad z_1 = f(t)e^{i\theta_1},$$
$$z_2 = f(t)e^{i\theta_1} + g(t)e^{i\theta_2}$$

where $f(0) = 1$, $f(1) = \varepsilon$
$g(0) = \varepsilon$, $g(1) = 1$.

then

$$S^1 \times S^1 \times \{0\} \longmapsto C_1$$

$$S^1 \times S^1 \times \{1\} \longmapsto C_3$$

So this cycle interpolates between the first and last terms in the Jacobi relation, but doesn't always lie in our configuration space: when $f(t) = g(t)$, we may have $z_1 = z_3$. We remove a nbhd of these points; the boundary of this nbhd is C_2. Hence $C_1 + C_2 + C_3 = 0$: the Jacobi relation is satisfied.

One can similarly prove the Poisson identity:

$$[a, bc] = [a,b]c + (-1)^{(|a|-1)|b|} b[a,c].$$

<u>Thm</u>: This is a complete set of generators and relations for the operad $\mathcal{P}$.

An algebra over $H_*(\mathrm{Conf}(\mathbb{C},k))$ is called a Gerstenhaber algebra.

There is an obvious map of operads

$$\mathrm{Conf}(\mathbb{C},k) \longrightarrow \overline{\mathcal{M}}_{0,k+1}$$

by regarding $\mathbb{C} = \mathbb{CP}^1 \setminus \infty$.
Thus, there is a map of operads

$$H_*(\mathrm{Conf}(\mathbb{C},k)) \to H_*(\overline{\mathcal{M}}_{0,k+1})$$

Hence, any algebra over the modular operad becomes a Gerstenhaber algebra, but the homology class corresponding to the Lie bracket maps to 0 $(H_1(\overline{\mathcal{M}}_{0,3}) \cong 0)$, so the Lie bracket is trivial.

We will see later some closed-string A-models which have non-trivial Lie bracket, but let us first mention some other interesting Gerstenhaber algebras.

E.g. <u>The closed-string B-model.</u>

M = complex mfd / algebraic variety.
One part of the closed-string B-model is the tangential cohomology:

$$HT^*(M) := H^*(M, \Lambda^* TM),$$

the cohomology of the sheaf of <u>polyvector fields</u>.

$\Lambda^* TM$ is a sheaf of Gerstenhaber algebras: the product is exterior product, and bracket is uniquely extended from the Lie bracket on TM together with

$$[f, v] = v(f).$$

This bracket is called the <u>Schouten-Nijenhuis bracket</u>. It can be written more explicitly as

$$[v_1 \dots v_m, w_1 \dots, w_n] = \sum_{i,j} (-1)^{i+j} [v_i, w_j] v_1 \dots \hat{v}_i \dots v_m w_1 \dots \hat{w}_j \dots w_n,$$

$\Rightarrow$ $[f, -] = \iota_{df}$.

It follows that $H^*(\Lambda^* TM)$ is a Gerstenhaber algebra.

<u>Defn</u>: A <u>Landau-Ginzburg model</u> (M, W) is a complex manifold M with a holomorphic function W on it.

Define

$$HT^*(M, W) := H^*\Big(H^*(M, \Lambda^* TM), [W, -] \Big)$$

Note this is still a Gerstenhaber algebra, as $[W, -]$ satisfies a Leibniz rule w.r.t. Lie bracket (Jacobi identity) and product (Poisson identity).

<u>Special case:</u> If $M = \Bbbk^n$, we get the Koszul complex of the partial derivatives of W:

$$HT^*(\Bbbk^n, W) \cong H^*\left(\Lambda^*\left(\Bbbk[z_1, \dots, z_n]\right), [W,-] = \iota_{dW}\right).$$

If M is mirror to $M^\vee$ (resp., $(M^\vee, W^\vee)$), then closed-string mirror symmetry predicts

$$QH^*_{small}(M) \cong HT^*(M^\vee) \quad \left(\text{resp. } HT^*(M^\vee, W^\vee)\right)$$

as algebras over Λ_0 (or $\Lambda = \Lambda_0[T^{-1}]$).

In general it is easier to prove mirror symmetry over Λ than over Λ_0; inverting T means we stay away from the singular point in the moduli space where $T = 0$.

E.g. $M = T^2$, $QH^*(M) \cong \Lambda^*(\Lambda^2)$ (no J-hol. spheres in T^2, so quantum cup product = classical cup product).
$M^\vee$ = elliptic curve$/\Lambda$

$$HT^*(M^\vee) \cong H^*(\mathcal{O} \oplus TM^\vee) \cong H^*(\mathcal{O} \oplus \mathcal{O})$$
$$\cong \Lambda^*(\Lambda^2).$$

E.g. $M = \mathbb{CP}^1$, $QH^*(M) \cong \Lambda[z]/_{z^2 = T}$

$$(M^\vee, W^\vee) = \mathrm{Spec}\left(\Lambda[z, z^{-1}], W^\vee = z + Tz^{-1}\right)$$

Actually, here we have convergence: can substitute actual number for T, and $M^\vee = \mathbb{C}^*$, $W^\vee = z + Tz^{-1}$.

$$HT^*(\mathbb{C}^*, W^\vee) \cong \mathbb{C}[z, z^{-1}]/z^2 = T \cong \mathbb{C}[z]/z^2 = T.$$

But there's something funny: on B-model we get a Gerstenhaber algebra, which may have nontrivial Lie bracket:

E.g. $M^\vee = \mathbb{C}^*$, $HT^*(M^\vee) \cong \mathbb{C}[z, z^{-1}, \partial_z]$, and

$$[z, \partial_z] = \partial_z(z) = 1.$$

So what could be mirror to this? (We'll see next time: <u>symplectic cohomology</u>)

Another question: what is the analogue of the 'big' quantum cohomology on the B-side? (we'll see later when we talk about <u>deformation theory</u>).
First, we'll see a reason why the Lie bracket vanishes on the B-model sometimes:

<u>Lem</u>: If M is Calabi-Yau (i.e. $c_1(TM) = 0$, or equivalently $\exists\ \Omega \in \Omega^{n,0}(M)$ non-vanishing holom. volume form) and compact Kähler, the Lie bracket vanishes.

<u>Pf</u>: $\lrcorner\,\Omega : \Lambda^p TM \xrightarrow{\sim} \Omega^{n-p} M$

$$\Rightarrow \lrcorner\,\Omega : (\Omega^{0,j}(\Lambda^p TM), \bar\partial) \xrightarrow{\sim} (\Omega^{n-p,j} M, \bar\partial).$$

This respects $\bar\partial$ (Ω holomorphic), and we let Δ denote the pullback of ∂.
Then

(*) $[\gamma_1, \gamma_2] = \pm \Delta(\gamma_1 \wedge \gamma_2) \pm \Delta\gamma_1 \wedge \gamma_2 \pm \gamma_1 \wedge \Delta\gamma_2$
(Tian-Todorov lemma).
Of course Δ vanishes on $\bar\partial$ cohomology (Hodge-de Rham spectral sequence degenerates at E_1), hence so does $[\,,\,]$.